The Finite Element Method on Domains with Conical or Angular Points

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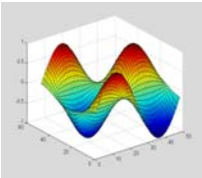
Introduction

The Finite Element Method (FEM) is an important method for the numerical approximation of solutions of Partial Differential Equations (PDEs).

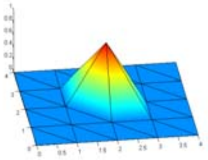


Approximation on the mesh

Most PDEs do not have explicit analytic solutions. Numerical approximation on a discretized domain is a powerful tool for studying the solutions of these PDEs. This may reveal interesting physical phenomena.

Starting from a domain decomposition, piecewise polynomials (continuous functions that are polynomials on each patch of the partition) are typically used to approximate the solution. A main goal in the FEM is to estimate and reduce the error between the numerical solution and the exact solution.

Approximation by piecewise polynomials on the domain results in a system of linear equations involving a large number of unknowns.



A basis function in 2-D

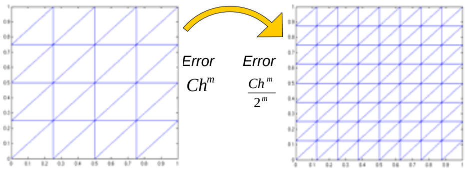
To fix ideas, we consider only the Laplace operator with mixed boundary conditions (Equation (1)) for $\Omega \subset R^2$. However, similar results extend to other elliptic equations for $\Omega \subset R^d, d = 2,3$.

$$(1) \quad -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial_D \Omega, \quad \partial u / \partial n = 0 \text{ on } \partial_N \Omega$$

Theorem 0. Let Ω be a bounded open set with smooth boundary, then if $u \in H^{m+1}(\Omega)$, $f \in H^{m-1}(\Omega)$,

$$\| u - u_h \|_{H^1(\Omega)} \leq Ch^m \, | u |_{H^{m+1}(\Omega)} \leq Ch^m \| f \|_{H^{m-1}(\Omega)},$$

where u_h stands for the FE approximation with piecewise polynomials of degree m, and h represents the largest diameter of each piece of the mesh.



Uniform mesh with diameter h — *Uniform mesh with diameter $h/2$*

Consequently, for a certain class of elliptic PDEs with solutions of full regularity, the error will be reduced in the rate of h^m as long as the largest diameter of each patch does not exceed h.

Problem Addressed

The geometry of the boundary and the change of boundary conditions will influence the regularity of the solution. Hence, on domains with non-smooth points (vertices in 2-D or vertices and edges in 3-D), the derivatives of the solution may increase dramatically near these points, so that the solution is not even in $H^2(\Omega)$, which leads to bad convergence rates of the numerical approximations (Figure 1).

0.92
0.84
0.79
0.75
0.72

Convergence rates

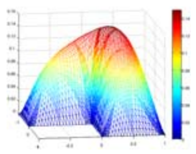


Figure 1. the solution of (1) with f=1 on the L-shape and the convergence rates of the numerical solutions on uniform meshes

Nevertheless, for such domains and change of boundary conditions, it is possible to recover the usual approximation results (Theorem 0) by using weighted Sobolev spaces. This then yields meshes that provide optimal rates of convergence.

From this point, we consider Equation (1) on a polygon P.

Goals:

- Well-posedness and regularity of the solution
- Approximation theory in weighted spaces
- Optimal meshes for the convergence rate
- Study of the multigrid method (MG) on graded meshes

Methodology

Define the weighted Sobolev space

$$\kappa_a^m(P) = \{ u \in L^2_{loc}(P), r^{|\alpha| - a} \partial_x^\alpha u \in L^2(P), | \alpha | \leq m \}$$

with norm

$$\| u \|^2_{\kappa_a^m(P)} := \Sigma_{|\alpha| \leq m} \| r^{|\alpha| - a} \partial_x^\alpha u \|^2_{L^2(P)} .$$

Start with the Mellin (Kondratiev) transformation on each "vertex" (corner without end), we derive that in the neighborhood C of the "vertex",

$$\Delta : \kappa_{a+1}^{m+1}(C) \bigcap \{ u |_{\partial_D C} = 0 \} \to \kappa_{a-1}^{m-1}(C)$$

is an isomorphism, if a is away from a set of countable values.

From the research on the weak solution near the "vertices", we obtain the uniqueness and existence of the solution on P.

Theorem 1. For a polygon P, there exists a constant $\eta > 0$, depending on P and on the problem, such that

$$\Delta : \kappa_{\varepsilon+1}^{m+1}(P) \bigcap \{ u |_{\partial_D P} = 0 \} \to \kappa_{\varepsilon-1}^{m-1}(P), m \geq 0$$

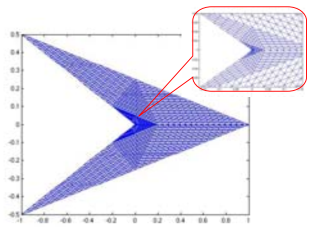
is an isomorphism for any $| \varepsilon | < \eta$.

To generate the optimal mesh and achieve a uniform error estimate, we apply the homogeneity of the weighted Sobolev spaces and discover that if the mesh sizes decay like k near the "vertices", for $k = 2^{-m/\varepsilon}, 0 < \varepsilon < \eta$, the classical result in Theorem 0 can be recovered for the FE solution.

Theorem 2. There exists a constant C, independent of the number of refinements, such that

$$\| u - u_n \|_{H^1(P)} \leq C \dim(V_n)^{-m/2} \| f \|_{H^{m-1}(P)}$$

for any $f \in \kappa_{\varepsilon-1}^{m-1}(P) \cap H^{m-1}(P)$, if the mesh size decays in ratio k, which is determined by ε, near the vertices, where $\dim(V_n)$ is the dimension of the finite subspace V_n spanned by piecewise polynomials on the mesh after n refinements.



Triangles decay near (0, 0) with ratio=0.2

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Numerical Result

Model problem:

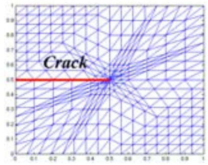
$$(2) \quad -\Delta u = 1 \text{ in } P, \quad u = 0 \text{ on } \partial P$$

where P is the unit square shown below. We solve this equation on the meshes with linear functions for $k = 0.1, 0.2, 0.3, 0.4, 0.5$. The convergence rates are listed in the following table.

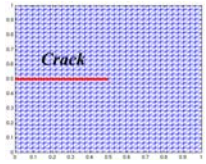
level j	e: k=0.1	e: k=0.2	e: k=0.3	e: k=0.4	e: k=0.5
3	0.76	0.79	0.79	0.83	0.77
4	0.88	0.90	0.89	0.82	0.76
5	0.94	0.95	0.91	0.79	0.70
6	0.97	0.97	0.92	0.76	0.63
7	0.99	0.98	0.91	0.73	0.57
8	0.99	0.98	0.91	0.71	0.54
9	1.00	0.99	0.90	0.69	0.52

$e = \log_2 [\frac{|u_j - u_{j-1}|_{1,P}}{|u_{j+1} - u_j|_{1,P}}]$ *represents the convergence rate; j stands for the level of refinements.*

This table convincingly verifies our theoretical prediction that if the triangles decay with the ratio less than 0.25 near the tip of the crack, the optimal rate of convergence can be obtained, as in Theorem 2.



Domain $P, k = 0.2$ mesh after 3 refinements



Domain $P, k = 0.5$ mesh after 4 refinements

The Multigrid Method (MG)

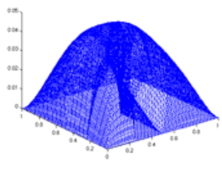
By using a space decomposition for elliptic projections and properties of weighted Sobolev spaces, the following estimate shows the multigrid V-cycle converges uniformly on the linear system of equations resulting from good graded meshes for corner-like singularities [3].

$$\| I - BA \|_a^2 = c_0 / (1 + c_0) \leq c_1 / (c_1 + c_2 n)$$

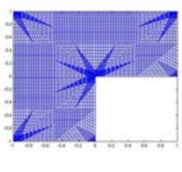
The numerical tests on graded meshes for the crack problem (2) provide solid support for our theory.

levels	2	3	4	5	6
ρ_{MG} (GS)	0.40	0.53	0.56	0.53	0.50

Convergence factors for the MG v(1,1)-cycle with GS smoother



Numerical solution for Equation (2)



Mesh for quadratics after 6 refinements, k=0.2

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[1] C.Bacuta, V.Nistor and L.Zikatanov, *Improving the rate of convergence of high order finite elements on polygons and domains with cusps*, 2005.
[2] H. Li, A. Mazzucato and V.Nistor, *On the analysis of the Finite Element Method on general polygonal domains I: a priori estimates*, 2007
[3] J. Brannick, H. Li and L. Zikatanov, *Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities*, 2007

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