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Multigrid methods for saddle point problems: Oseen system

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Dedicated to Professor Ivo Babuška on the Occasion of his Ninetieth Birthday.

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ABSTRACT

We develop and analyze multigrid methods for the Oseen system in fluid flow. We show that the W-cycle algorithm is a uniform contraction if the number of smoothing steps is sufficiently large. Numerical results that illustrate the performance of the methods are also presented.

1. Introduction

Let Ω be a polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ and $L_2^0(\Omega) = \{q \in L_2(\Omega) : \int_\Omega q\,dx = 0\}$. The Oseen system with the no-slip boundary condition is to find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx \qquad \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d, \tag{1.1a}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall\, q \in L_2^0(\Omega), \tag{1.1b}$$

where

$$a(\boldsymbol{w}, \boldsymbol{v}) = \nu \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v}\,dx + \int_\Omega (\boldsymbol{\varrho} \cdot \nabla \boldsymbol{w}) \cdot \boldsymbol{v}\,dx \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in [H_0^1(\Omega)]^d, \tag{1.2}$$

$$b(\boldsymbol{v}, q) = \int_\Omega (\nabla \cdot \boldsymbol{v}) q\,dx \qquad \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d,\ q \in L_2^0(\Omega). \tag{1.3}$$

Throughout the paper we will follow standard notation for differential operators, function spaces and norms that can be found for example in [1,2].

Here $\boldsymbol{u}$ is the fluid velocity, p is the pressure, ν is the kinematic viscosity, $\boldsymbol{f} \in [L_2(\Omega)]^d$ is the body force density and $\boldsymbol{\varrho} \in [W_\infty^1(\Omega)]^d \cap H(\mathrm{div}^0; \Omega)$ is a wind function. Such a system arises for example from a fixed point iteration for the stationary Navier–Stokes system [3].

Our goal is to extend the multigrid methods in [4] for the Stokes system (where $\boldsymbol{\varrho} = \mathbf{0}$) to the nonsymmetric Oseen system (1.1). The key idea is to treat the Oseen system and its adjoint problem simultaneously. Then all the results in [4] remain

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valid after minor modifications. Since the numerical scheme in [4] is not designed for flows with high Reynolds numbers, we will only consider the case where $\nu = 1$.

There is a large literature on the numerical studies of multigrid methods for the Stokes and Oseen systems (cf. [5–8] and the references therein). On the other hand there are relatively few papers on the convergence of multigrid methods for these systems [9–15]. The convergence results in these papers are established with respect to norms different from the energy norm $|\cdot|_{H^1(\Omega)} \times \|\cdot\|_{L_2(\Omega)}$, and the analyses also require Ω to be convex. Moreover the contraction number bounds in some of these papers are $O(1/\sqrt{m})$, where m is the number of smoothing steps. In this paper (and [4]) we are able to prove uniform convergence of the W-cycle algorithm in the energy norm on general domains, with an $O(1/m)$ contraction number estimate for convex domains.

The rest of the papers is organized as follows. Our multigrid method is based on a mixed finite element discretization of (1.1), which is discussed in Section 2. The multigrid algorithms are introduced in Sections 3 and 4, followed by a convergence analysis in Section 5. Numerical results for two dimensional domains are presented in Section 6 and we end with some concluding remarks in Section 7.

All the constants in the paper are independent of the mesh levels. To avoid the proliferation of constants, we will also use the notation $A \lesssim B$ to represent the statement $A \leq (\text{constant}) \times B$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

2. A mixed finite element method

The compact form of (1.1) is to find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$\mathcal{B}\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall (\boldsymbol{v}, q) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega), \tag{2.1}$$

where the bilinear form $\mathcal{B}(\cdot,\cdot)$ is defined by

$$\mathcal{B}\big((\boldsymbol{w}, r), (\boldsymbol{v}, q)\big) = a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{v}, r) + b(\boldsymbol{w}, q). \tag{2.2}$$

Given $\boldsymbol{f}^* \in [L_2(\Omega)]^2$, the adjoint problem for the Oseen system is to find $(\boldsymbol{u}^*, p^*) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{u}^*, p^*)\big) = \int_\Omega \boldsymbol{f}^* \cdot \boldsymbol{v}\, dx \quad \forall (\boldsymbol{v}, q) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega). \tag{2.3}$$

Let $\mathcal{T}_h$ be a quasi-uniform simplicial triangulation of Ω and $V_h \times Q_h$ be the P_2-P_1 Taylor–Hood [16] finite element subspace of $[H_0^1(\Omega)]^d \times L_2^0(\Omega)$ associated with $\mathcal{T}_h$. The discrete problem for (2.1) is to find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall (\boldsymbol{v}, q) \in V_h \times Q_h, \tag{2.4}$$

and the discrete problem for (2.3) is to find $(\boldsymbol{u}_h^*, p_h^*) \in V_h \times Q_h$ such that

$$\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{u}_h^*, p_h^*)\big) = \int_\Omega \boldsymbol{f}^* \cdot \boldsymbol{v}\, dx \quad \forall (\boldsymbol{v}, q) \in V_h \times Q_h. \tag{2.5}$$

It is well-known [17–19] that

$$\inf_{0 \neq q \in Q_h} \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in V_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)} \|q\|_{L_2(\Omega)}} \geq \beta > 0, \tag{2.6}$$

where the constant β is independent of the mesh parameter h.

Since the divergence-free assumption on the wind function $\boldsymbol{\varrho}$ implies that

$$a(\boldsymbol{v}, \boldsymbol{v}) = \int_\Omega |\nabla \boldsymbol{v}|^2\, dx \quad \forall \boldsymbol{v} \in [H_0^1(\Omega)]^d, \tag{2.7}$$

the stability of the saddle problems (2.4) and (2.5) can be established by the standard theory [20–22].

In particular, we have

$$\sup_{(\boldsymbol{0},0) \neq (\boldsymbol{w}, r) \in V_h \times Q_h} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{|\boldsymbol{w}|_{H^1(\Omega)} + \|r\|_{L_2(\Omega)}} \approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \forall (\boldsymbol{v}, q) \in V_h \times Q_h, \tag{2.8}$$

$$\sup_{(\boldsymbol{0},0) \neq (\boldsymbol{w}, r) \in V_h \times Q_h} \frac{\mathcal{B}\big((\boldsymbol{w}, r), (\boldsymbol{v}, q)\big)}{|\boldsymbol{w}|_{H^1(\Omega)} + \|r\|_{L_2(\Omega)}} \approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \forall (\boldsymbol{v}, q) \in V_h \times Q_h. \tag{2.9}$$

Since the wind function $\boldsymbol{\varrho}$ belongs to $[W_\infty^1(\Omega)]^d$ and $\boldsymbol{u}$ belongs to $[H_0^1(\Omega)]^d$, the vector field $\boldsymbol{\varrho} \cdot \nabla \boldsymbol{u}$ belongs to $[L_2(\Omega)]^d$. Therefore the solution of (2.1) is also the solution of the Stokes system with body force density $\boldsymbol{f} - \boldsymbol{\varrho} \cdot \nabla \boldsymbol{u}$ and hence it shares the same elliptic regularity as the solutions of the Stokes system. The same also holds for the solution $\boldsymbol{u}^*$ of (2.3). Therefore we have the following discretization error estimates:

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)}+\|p-p_h\|_{L_2(\Omega)}\leq Ch^\alpha\|\boldsymbol{f}\|_{H^{-1+\alpha}(\Omega)}, \tag{2.10}$$

$$\|\boldsymbol{u}^*-\boldsymbol{u}_h^*\|_{H^1(\Omega)}+\|p^*-p_h^*\|_{L_2(\Omega)}\leq Ch^\alpha\|\boldsymbol{f}^*\|_{H^{-1+\alpha}(\Omega)}, \tag{2.11}$$

where $\alpha\in(1/2,1]$ is the index of elliptic regularity for the Stokes system on Ω and $\alpha=1$ if Ω is convex. (The elliptic regularity theory for the Stokes problem on polyhedral domains can be found for example in [23–27].)

3. Multigrid set-up

Let $\mathcal{T}_0$ be an initial triangulation of Ω and the triangulation $\mathcal{T}_k$ $(k\geq 1)$ be generated from $\mathcal{T}_{k-1}$ through uniform refinement. Let $V_k\subset[H_0^1(\Omega)]^d$ (respectively $Q_k\subset L_2^0(\Omega)$) be the continuous P_2 (respectively P_1) Lagrange finite element space associated with $\mathcal{T}_k$. The kth level discrete problem for (2.1) is to find $(\boldsymbol{u}_k,p_k)\in V_k\times Q_k$ such that

$$\mathcal{B}\big((\boldsymbol{u}_k,p_k),(\boldsymbol{v},q)\big)=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\quad\forall(\boldsymbol{v},q)\in V_k\times Q_k. \tag{3.1}$$

3.1. The mesh-dependent inner product $[\cdot,\cdot]_k$

In order to define and analyze the multigrid algorithms in terms of operators, we introduce a mesh-dependent inner product $[\cdot,\cdot]_k$ on $V_k\times Q_k$ as follows.

Let $\mathcal{N}_{k,2}$ (respectively $\mathcal{N}_{k,1}$) be the set of the nodes of the P_2 (respectively P_1) Lagrange finite element space associated with $\mathcal{T}_k$. The mesh-dependent inner product $((\cdot,\cdot))_k$ on V_k and the mesh-dependent inner product $(\cdot,\cdot)_k$ on Q_k are given by

$$((\boldsymbol{v},\boldsymbol{w}))_k=h_k^d\sum_{x\in\mathcal{N}_{k,2}}\boldsymbol{v}(x)\cdot\boldsymbol{w}(x)\qquad\forall\,\boldsymbol{v},\boldsymbol{w}\in V_k, \tag{3.2}$$

$$(q,r)_k=h_k^{d+2}\sum_{x\in\mathcal{N}_{k,1}}q(x)r(x)\qquad\forall\,q,r\in Q_k. \tag{3.3}$$

Note that we have

$$((\boldsymbol{v},\boldsymbol{v}))_k\approx\|\boldsymbol{v}\|_{L_2(\Omega)}^2\qquad\forall\,\boldsymbol{v}\in V_k, \tag{3.4}$$

$$(q,q)_k\approx h_k^2\|q\|_{L_2(\Omega)}^2\qquad\forall\,q\in Q_k. \tag{3.5}$$

Remark 3.1. The scalings for $((\cdot,\cdot))_k$ and $(\cdot,\cdot)_k$ are different because second order derivatives for the fluid velocity appear in the strong form of the Oseen system, while only first order derivatives of the pressure appear there.

We then define

$$[(\boldsymbol{v},q),(\boldsymbol{w},r)]_k=((\boldsymbol{v},\boldsymbol{w}))_k+(q,r)_k\quad\forall(\boldsymbol{v},q),(\boldsymbol{w},r)\in V_k\times Q_k. \tag{3.6}$$

3.2. The operator $\mathbb{B}_k$

The operator $\mathbb{B}_k:V_k\times Q_k\longrightarrow V_k\times Q_k$ is defined by

$$[\mathbb{B}_k(\boldsymbol{w},r),(\boldsymbol{v},q)]_k=\mathcal{B}\big((\boldsymbol{w},r),(\boldsymbol{v},q)\big)\qquad\forall(\boldsymbol{v},q),(\boldsymbol{w},r)\in V_k\times Q_k. \tag{3.7}$$

The transpose $\mathbb{B}_k^t$ of $\mathbb{B}_k$ with respect to $[\cdot,\cdot]_k$ satisfies

$$[\mathbb{B}_k^t(\boldsymbol{w},r),(\boldsymbol{v},q)]_k=[(\boldsymbol{w},r),\mathbb{B}_k(\boldsymbol{v},q)]_k=[\mathbb{B}_k(\boldsymbol{v},q),(\boldsymbol{w},r)]_k=\mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big) \tag{3.8}$$

for all $(\boldsymbol{v},q),(\boldsymbol{w},r)\in V_k\times Q_k$.

We can now rewrite (3.1) as

$$\mathbb{B}_k(\boldsymbol{u}_k,p_k)=(\boldsymbol{f}_k,0), \tag{3.9}$$

where $\boldsymbol{f}_k\in V_k$ is defined by $((\boldsymbol{f}_k,\boldsymbol{v}))_k=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,dx\;\forall\,\boldsymbol{v}\in V_k$.

Our goal is to design multigrid methods for equations of the form

$$\mathbb{B}_k(\boldsymbol{v},q)=(\boldsymbol{g},z) \tag{3.10}$$

that include (3.9) as a special case. For this purpose it is useful to also consider multigrid methods for equations of the form

$$\mathbb{B}_k^t(\boldsymbol{v},q)=(\boldsymbol{g},z). \tag{3.11}$$

Note that the kth level discrete problem for (2.3) is a special case of (3.11) because of (3.8).

3.3. The operator $\mathbb{S}_k$

Here we recall the construction of an operator $\mathbb{S}_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ that plays a key role in the multigrid algorithms for the Stokes system in [4].

Let the discrete Laplacian $\Delta_k : V_k \longrightarrow V_k$ be defined by

$$((-\Delta_k \boldsymbol{v}, \boldsymbol{w}))_k = \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w}\, dx \quad \forall\, \boldsymbol{v}, \boldsymbol{w} \in V_k, \tag{3.12}$$

and $L_k : V_k \longrightarrow V_k$ be a symmetric positive definite (SPD) operator with respect to $((\cdot,\cdot))_k$ such that

$$\kappa_1 \le \lambda_{\min}\big(L_k(-\Delta_k)\big) \le \lambda_{\max}\big(L_k(-\Delta_k)\big) \le \kappa_2. \tag{3.13}$$

Remark 3.2. The estimate (3.13) means that the operator L_k is a (uniform) optimal preconditioner for $-\Delta_k$. Such an operator can be constructed by multigrid [28] or domain decomposition [29]. Note that (3.12) and (3.13) imply

$$((L_k^{-1}\boldsymbol{v}, \boldsymbol{v}))_k \approx |\boldsymbol{v}|^2_{H^1(\Omega)} \quad \forall\, \boldsymbol{v} \in V_k. \tag{3.14}$$

The operator $\mathbb{S}_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ is then defined by

$$\mathbb{S}_k(\boldsymbol{v}, q) = (L_k \boldsymbol{v}, h_k^2 q). \tag{3.15}$$

Clearly $\mathbb{S}_k$ is SPD with respect to $[\cdot,\cdot]_k$. Moreover, by construction we have

$$\begin{aligned}
[\mathbb{S}_k^{-1}(\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k &= [(L_k^{-1}\boldsymbol{v}, h_k^{-2} q), (\boldsymbol{v}, q)]_k \\
&= ((L_k^{-1}\boldsymbol{v}, \boldsymbol{v}))_k + h_k^{-2}(q, q)_k \\
&\approx |\boldsymbol{v}|^2_{H^1(\Omega)} + \|q\|^2_{L_2(\Omega)} \quad \forall (\boldsymbol{v}, q) \in V_k \times Q_k,
\end{aligned} \tag{3.16}$$

where we have also used (3.5) and (3.14).

Remark 3.3. Block diagonal preconditioners similar to $\mathbb{S}_k$ (cf. [30,31]) can also be used to solve the Oseen system by the preconditioned GMRES algorithm [32,33].

3.4. The operators $\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k$ and $\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t$

The properties of $\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k$ and $\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t$ established in the following lemma will play a key role in the analysis of the smoothers.

Lemma 3.4. *We have*

$$[\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k^{\frac{1}{2}} \approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{v} \in V_k,\ q \in Q_k, \tag{3.17}$$

$$[\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t(\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k^{\frac{1}{2}} \approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{v} \in V_k,\ q \in Q_k. \tag{3.18}$$

Proof. We can establish the equivalence (3.17) by using (2.8), (3.7), (3.16) and duality as follows:

$$\begin{aligned}
[\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k^{\frac{1}{2}} &= [\mathbb{S}_k^{-1}\mathbb{S}_k\mathbb{B}_k(\boldsymbol{v}, q), \mathbb{S}_k\mathbb{B}_k(\boldsymbol{v}, q)]_k^{\frac{1}{2}} \\
&= \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{[\mathbb{S}_k^{-1}\mathbb{S}_k\mathbb{B}_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)]_k}{[\mathbb{S}_k^{-1}(\boldsymbol{w}, r), (\boldsymbol{w}, r)]_k^{\frac{1}{2}}} \\
&= \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{[\mathbb{S}_k^{-1}(\boldsymbol{w}, r), (\boldsymbol{w}, r)]_k^{\frac{1}{2}}} \\
&\approx \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{|\boldsymbol{w}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}} \\
&\approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}.
\end{aligned}$$

Similarly the equivalence (3.17) follows from (2.9), (3.8) and (3.16). □

In view of (3.4)–(3.6) and a standard inverse estimate, we immediately have the following corollary on the spectral radii $\rho(\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k)$ and $\rho(\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t)$.

Corollary 3.5. *There exists a positive constant C independent of k such that*

$$\rho(\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k),\ \rho(\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t) \le C h_k^{-2}. \tag{3.19}$$

4. Multigrid algorithms

We are now ready to introduce the multigrid algorithms for (3.10) and (3.11). We begin with the two main ingredients: (i) intergrid transfer operators that move information between consecutive grids, and (ii) smoothing operators that can damp out the highly oscillatory part of the error.

4.1. Intergrid transfer operators

Since the finite element spaces are nested, i.e., $V_{k-1} \times Q_{k-1} \subset V_k \times Q_k$ for $k \geq 1$, we can simply take the coarse-to-fine operator $I_{k-1}^k : V_{k-1} \times Q_{k-1} \longrightarrow V_k \times Q_k$ to be the natural injection. Then the fine-to-coarse operator $I_k^{k-1} : V_k \times Q_k \longrightarrow V_{k-1} \times Q_{k-1}$ is the transpose of I_{k-1}^k with respect to the mesh-dependent inner products, i.e.,

$$[I_k^{k-1}(\boldsymbol{v}, q), (\boldsymbol{w}, r)]_{k-1} = [(\boldsymbol{v}, q), I_{k-1}^k(\boldsymbol{w}, r)]_k \tag{4.1}$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$ and $(\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$.

4.2. Smoothers

The smoothers for the multigrid algorithms are modifications of the ones in [4] for the Stokes system. The key ingredient is the operator $\mathbb{S}_k$ introduced in Section 3.3.

4.2.1. Post-smoothers

The post-smoother for (3.10) is given by

$$(\boldsymbol{v}_{\rm new}, q_{\rm new}) = (\boldsymbol{v}_{\rm old}, q_{\rm old}) + \delta_k \mathbb{B}_k^t \mathbb{S}_k \big((\boldsymbol{g}, z) - \mathbb{B}_k(\boldsymbol{v}_{\rm old}, q_{\rm old})\big), \tag{4.2}$$

where $\delta_k > 0$ is a damping factor determined by the condition that the spectral radius $\rho(\delta_k \mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k)$ satisfies

$$\rho(\delta_k \mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k) \leq 1. \tag{4.3}$$

In view of (3.19), we can take $\delta_k = Ch_k^2$ for some positive constant C independent of k.

The error propagation operator for this smoother is given by

$$R_k = Id_k - \delta_k \mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k, \tag{4.4}$$

where Id_k is the identity operator on $V_k \times Q_k$.

Remark 4.1. The definition of (4.2) is motivated by the fact that it is the Richardson iteration for the SPD problem

$$\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k(\boldsymbol{v}, q) = \mathbb{B}_k^t \mathbb{S}_k(\boldsymbol{g}, z), \tag{4.5}$$

which is equivalent to (3.10). Note that (3.19) indicates that $\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k$ behaves like the discrete operator of a second order SPD elliptic problem and hence we can expect the usual smoothing property for R_k.

Similarly the post-smoother for (3.11) is given by

$$(\boldsymbol{v}_{\rm new}, q_{\rm new}) = (\boldsymbol{v}_{\rm old}, q_{\rm old}) + \delta_k \mathbb{B}_k \mathbb{S}_k \big((\boldsymbol{g}, z) - \mathbb{B}_k^t(\boldsymbol{v}_{\rm old}, q_{\rm old})\big), \tag{4.6}$$

and its error propagation operator is given by

$$\tilde{R}_k = Id_k - \delta_k \mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t. \tag{4.7}$$

Again $\delta_k = Ch_k^2$ is chosen so that

$$\rho(\delta_k \mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t) \leq 1. \tag{4.8}$$

4.2.2. Pre-smoothers

The pre-smoother for (3.10) is given by

$$(\boldsymbol{v}_{\rm new}, q_{\rm new}) = (\boldsymbol{v}_{\rm old}, q_{\rm old}) + \delta_k \mathbb{S}_k \mathbb{B}_k^t \big((\boldsymbol{g}, z) - \mathbb{B}_k(\boldsymbol{v}_{\rm old}, q_{\rm old})\big), \tag{4.9}$$

and its error propagation operator is given by

$$S_k = Id_k - \delta_k \mathbb{S}_k \mathbb{B}_k^t \mathbb{B}_k. \tag{4.10}$$

Similarly the pre-smoother for (3.11) is given by

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + \delta_k \mathbb{S}_k \mathbb{B}_k\big((\boldsymbol{g}, z) - \mathbb{B}_k^t(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big), \tag{4.11}$$

with the corresponding error propagation operator

$$\tilde{S}_k = Id_k - \delta_k \mathbb{S}_k \mathbb{B}_k \mathbb{B}_k^t. \tag{4.12}$$

Remark 4.2. The definitions of the pre-smoothers are motivated by the relations

$$\mathcal{B}\big(S_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), \tilde{R}_k(\boldsymbol{w}, r)\big) \qquad \forall (\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k, \tag{4.13}$$

$$\mathcal{B}\big(R_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), \tilde{S}_k(\boldsymbol{w}, r)\big) \qquad \forall (\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k, \tag{4.14}$$

that follow from (3.7), (3.8), (4.4), (4.7), (4.10) and (4.12). These relations will allow us to obtain properties of the pre-smoothers from the corresponding properties of the post-smoothers through duality (cf. (5.22) below).

4.3. The multigrid algorithms

Let the output of the W-cycle algorithm for (3.1) with initial guess $(\boldsymbol{v}_0, q_0) \in V_k \times Q_k$ and m_1 (respectively m_2) pre-smoothing (respectively post-smoothing) steps be denoted by $MG_W(k, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2)$.

For $k = 0$, we take $MG_W(0, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2)$ to be $\mathbb{B}_0^{-1}(\boldsymbol{g}, z)$.

For $k \geq 1$, we compute $MG_W(k, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2)$ recursively in three steps.

- *Pre-smoothing* Apply the iteration defined by (4.9) m_1 times with initial guess $(\boldsymbol{v}_0, q_0)$ to obtain $(\boldsymbol{v}_{m_1}, q_{m_1})$.
- *Coarse Grid Correction* Let $(\boldsymbol{g}', z\prime) = I_k^{k-1}\big((\boldsymbol{g}, z) - \mathbb{B}_k(\boldsymbol{v}_{m_1}, q_{m_1})\big)$ be the transferred residual of $(\boldsymbol{v}_{m_1}, q_{m_1})$. We compute $(\boldsymbol{v}'_1, q'_1), (\boldsymbol{v}'_2, q'_2) \in V_{k-1} \times Q_{k-1}$ by
$$(\boldsymbol{v}'_1, q'_1) = MG_W(k-1, (\boldsymbol{g}', z\prime), (\boldsymbol{0}, 0), m_1, m_2),$$
$$(\boldsymbol{v}'_2, q'_2) = MG_W(k-1, (\boldsymbol{g}', z\prime), (\boldsymbol{v}'_1, q'_1), m_1, m_2),$$
and take $(\boldsymbol{v}_{m_1+1}, q_{m_1+1})$ to be $(\boldsymbol{v}_{m_1}, q_{m_1}) + I_{k-1}^k(\boldsymbol{v}'_2, q'_2)$.
- *Post-smoothing* Apply the iteration defined by (4.2) m_2 times with initial guess $(\boldsymbol{v}_{m_1+1}, q_{m_1+1})$ to obtain $(\boldsymbol{v}_{m_1+m_2+1}, q_{m_1+m_2+1})$.

The final output is $MG_W(k, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2) = (\boldsymbol{v}_{m_1+m_2+1}, q_{m_1+m_2+1})$.

We denote by $MG_V(k, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2)$ the output of the V-cycle algorithm for (3.10) with initial guess $(\boldsymbol{v}_0, q_0) \in V_k \times Q_k$ and m_1 (respectively m_2) pre-smoothing (respectively post-smoothing) steps. The computation of $MG_V(k, (\boldsymbol{g}, z), (\boldsymbol{v}_0, q_0), m_1, m_2)$ is identical with the computation for the W-cycle algorithm except in the coarse grid correction step, where we compute

$$(\boldsymbol{v}'_1, q'_1) = MG_V(k-1, (\boldsymbol{g}', z\prime), (\boldsymbol{0}, 0), m_1, m_2)$$

and take $(\boldsymbol{v}_{m_1+1}, q_{m_1+1})$ to be $(\boldsymbol{v}_{m_1}, q_{m_1}) + I_{k-1}^k(\boldsymbol{v}'_1, q'_1)$.

Remark 4.3. We will only analyze the W-cycle algorithm. But numerical results in Section 6 indicate that the V-cycle algorithm also converges.

Similarly we can define the W-cycle and V-cycle algorithms for (3.11).

4.4. Error propagation operators

Let the Riesz projection operator $P_k^{k-1} : V_k \times Q_k \longrightarrow V_{k-1} \times Q_{k-1}$ be the transpose of the coarse-to-fine operator with respect to the bilinear form $\mathcal{B}(\cdot, \cdot)$, i.e.,

$$\mathcal{B}\big(P_k^{k-1}(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), I_{k-1}^k(\boldsymbol{w}, r)\big) \tag{4.15}$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$ and $(\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$.

Let $E_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ be the error propagation operator for the W-cycle (respectively V-cycle) multigrid algorithm for (3.10). We have a well-known recursive relation [28,34,35]:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^p P_k^{k-1}) S_k^{m_1}, \tag{4.16}$$

where $p = 2$ (respectively $p = 1$) for the W-cycle (respectively V-cycle) algorithm, and the operator R_k and S_k are defined in (4.4) and (4.10) respectively.

The operator $\tilde{P}_k^{k-1} : V_k \times Q_k \longrightarrow V_{k-1} \times Q_{k-1}$ (the counterpart of P_k^{k-1} for problem (3.11)) is defined by

$$\mathcal{B}\big((\boldsymbol{w}, r), \tilde{P}_k^{k-1}(\boldsymbol{v}, q)\big) = \mathcal{B}\big(I_{k-1}^k(\boldsymbol{w}, r), (\boldsymbol{v}, q)\big) \tag{4.17}$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$ and $(\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$.

Remark 4.4. Since the coarse-to-fine operator I_{k-1}^k is the natural injection, we have

$$P_k^{k-1} I_{k-1}^k = Id_k = \tilde{P}_k^{k-1} I_{k-1}^k, \tag{4.18}$$
$$\mathcal{B}\big((Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = 0, \tag{4.19}$$
$$\mathcal{B}\big((\boldsymbol{w}, r), (Id_k - I_{k-1}^k \tilde{P}_k^{k-1})(\boldsymbol{v}, q)\big) = 0, \tag{4.20}$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$ and $(\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$, and

$$\mathcal{B}\big(P_k^{k-1}(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), \tilde{P}_k^{k-1}(\boldsymbol{w}, r)\big) \quad \forall (\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k. \tag{4.21}$$

5. Convergence analysis

We will use mesh-dependent norms to describe the smoothing and approximation properties that are used in the convergence analysis.

5.1. Mesh-dependent norms

Since $\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k$ is SPD with respect to the mesh-dependent inner product $[\cdot, \cdot]_k$, we can define a scale of mesh-dependent norms $\|\cdot\|_{s,k}$ $(0 \le s \le 1)$ by

$$\|(\boldsymbol{v}, q)\|_{s,k} = [(\mathbb{B}_k^t \mathbb{S}_k \mathbb{B}_k)^s (\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k^{\frac{1}{2}} \quad \forall (\boldsymbol{v}, q) \in V_k \times Q_k. \tag{5.1}$$

In view of (3.4)–(3.6) and (3.17), we have

$$\|(\boldsymbol{v}, q)\|_{0,k}^2 \approx \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + h_k^2 \|q\|_{L_2(\Omega)}^2 \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.2}$$
$$\|(\boldsymbol{v}, q)\|_{1,k}^2 \approx |\boldsymbol{v}|_{H^1(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2 \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k. \tag{5.3}$$

It follows from (5.2), (5.3) and the interpolation theory of Sobolev spaces [36,37] that

$$\|(\boldsymbol{v}, q)\|_{1-\alpha,k} \approx \|\boldsymbol{v}\|_{H^{1-\alpha}(\Omega)} + h_k^\alpha \|q\|_{L_2(\Omega)} \quad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.4}$$

where $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity in (2.10) and (2.11). Details can be found in the proof of [4, Lemma 4.4].

Remark 5.1. Since the norms for the discrete pressure only involve the $L_2(\Omega)$ norm with different scalings, all the estimates involving the discrete pressures become trivial in the multigrid analysis.

Note that (2.8) and (5.3) imply

$$\|(\boldsymbol{v}, q)\|_{1,k} \approx \sup_{(\mathbf{0},0) \neq (\boldsymbol{w}, r) \in V_k \times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|(\boldsymbol{w}, r)\|_{1,k}} \quad \forall (\boldsymbol{v}, q) \in V_k \times Q_k. \tag{5.5}$$

Similarly we define the mesh-dependent norms $\|\cdot\|_{\widetilde{s,k}}$ for $0 \le s \le 1$ by

$$\|(\boldsymbol{v}, q)\|_{\widetilde{s,k}} = [(\mathbb{B}_k \mathbb{S}_k \mathbb{B}_k^t)^s (\boldsymbol{v}, q), (\boldsymbol{v}, q)]_k^{\frac{1}{2}} \quad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.6}$$

and again we have the equivalences

$$\|(\boldsymbol{v}, q)\|_{\widetilde{1,k}} \approx |\boldsymbol{v}|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.7}$$
$$\|(\boldsymbol{v}, q)\|_{\widetilde{1,k}} \approx \sup_{(\mathbf{0},0) \neq (\boldsymbol{w}, r) \in V_k \times Q_k} \frac{\mathcal{B}\big((\boldsymbol{w}, r), (\boldsymbol{v}, q)\big)}{\|(\boldsymbol{w}, r)\|_{\widetilde{1,k}}} \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.8}$$
$$\|(\boldsymbol{v}, q)\|_{\widetilde{1-\alpha,k}} \approx \|\boldsymbol{v}\|_{H^{1-\alpha}(\Omega)} + h_k^\alpha \|q\|_{L_2(\Omega)} \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.9}$$

that follow from (2.9), (3.4)–(3.6) and (3.18).

The following result shows that P_k^{k-1} (respectively $\tilde{P}_k^{k-1}$) is stable with respect to $\|\cdot\|_{1,k}$ (respectively $\|\cdot\|_{\widetilde{1,k}}$).

Lemma 5.2. *We have*

$$\|P_k^{k-1}(\boldsymbol{v}, q)\|_{1,k-1} \lesssim \|(\boldsymbol{v}, q)\|_{1,k} \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k, \tag{5.10}$$
$$\|\tilde{P}_k^{k-1}(\boldsymbol{v}, q)\|_{\widetilde{1,k-1}} \lesssim \|(\boldsymbol{v}, q)\|_{\widetilde{1,k}} \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k. \tag{5.11}$$

Proof. Since $I_{k-1}^k : V_{k-1} \times Q_{k-1} \longrightarrow V_k \times Q_k$ is the natural injection, we have

$$\begin{aligned}\|P_k^{k-1}(\boldsymbol{v}, q)\|_{1,k-1} &\approx \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_{k-1}\times Q_{k-1}} \frac{\mathcal{B}\big(P_k^{k-1}(\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{|\boldsymbol{w}|_{H^1(\Omega)}+\|r\|_{L_2(\Omega)}}\\ &= \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_{k-1}\times Q_{k-1}} \frac{\mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{|\boldsymbol{w}|_{H^1(\Omega)}+\|r\|_{L_2(\Omega)}}\\ &\lesssim \|(\boldsymbol{v},q)\|_{1,k} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k,\end{aligned}$$

by (5.3) and (5.5).

The proof of (5.11) is similar. □

5.2. Smoothing and approximation properties

In view of (3.19), (5.1), (5.3), (5.6) and (5.7), we immediately have the following standard smoothing properties (cf. [34, Section 6.2.3.1] and [2, Section 6.5]):

$$\|R_k^m(\boldsymbol{v},q)\|_{1,k} \lesssim h_k^{-\tau} m^{-\tau/2}\|(\boldsymbol{v},q)\|_{1-\tau,k} \qquad \forall(\boldsymbol{v},q)\in V_k\times Q_k,\ 0\le\tau\le 1, \tag{5.12}$$

$$\|\tilde{R}_k^m(\boldsymbol{v},q)\|_{1,k}^{\sim} \lesssim h_k^{-\tau} m^{-\tau/2}\|(\boldsymbol{v},q)\|_{1-\tau,k}^{\sim} \qquad \forall(\boldsymbol{v},q)\in V_k\times Q_k,\ 0\le\tau\le 1. \tag{5.13}$$

A modification of the arguments for [4, Lemma 5.4] yields the following approximation properties.

Lemma 5.3. *For $k \ge 1$, we have the following approximation properties :*

$$\|(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q)\|_{1-\alpha,k} \lesssim h_k^{\alpha}\|(\boldsymbol{v},q)\|_{1,k} \qquad \forall(\boldsymbol{v},q)\in V_k\times Q_k, \tag{5.14}$$

$$\|(Id_k - I_{k-1}^k \tilde{P}_k^{k-1})(\boldsymbol{v},q)\|_{1-\alpha,k}^{\sim} \lesssim h_k^{\alpha}\|(\boldsymbol{v},q)\|_{1,k}^{\sim} \qquad \forall(\boldsymbol{v},q)\in V_k\times Q_k, \tag{5.15}$$

where $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity in (2.10) and (2.11).

Proof. We will focus on (5.14) since the proof of (5.15) is completely analogous.

Let $(\boldsymbol{v}, q) \in V_k \times Q_k$ be arbitrary and

$$(\boldsymbol{\zeta},\mu) = (Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q). \tag{5.16}$$

In view of (5.4), we only need to show that

$$\|\boldsymbol{\zeta}\|_{H^{1-\alpha}(\Omega)} + h_k^{\alpha}\|\mu\|_{L_2(\Omega)} \lesssim h_k^{\alpha}\|(\boldsymbol{v},q)\|_{1,k},$$

and the estimate for μ follows immediately from (5.3) and (5.10):

$$\|\mu\|_{L_2(\Omega)} \lesssim \|(\boldsymbol{\zeta},\mu)\|_{1,k} = \|(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q)\|_{1,k} \lesssim \|(\boldsymbol{v},q)\|_{1,k}.$$

We will prove the estimate for $\boldsymbol{\zeta}$ by a duality argument. Let $\boldsymbol{\chi} \in [H^{-1+\alpha}(\Omega)]^d$ be arbitrary, $(\boldsymbol{\xi},\theta) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ satisfy

$$\mathcal{B}\big((\boldsymbol{w},r),(\boldsymbol{\xi},\theta)\big) = \langle\boldsymbol{\chi},\boldsymbol{w}\rangle \quad \forall(\boldsymbol{w},r)\in[H_0^1(\Omega)]^d\times L_2^0(\Omega), \tag{5.17}$$

and $(\boldsymbol{\xi}_{k-1},\theta_{k-1}) \in V_{k-1}\times Q_{k-1}$ satisfy

$$\mathcal{B}\big((\boldsymbol{w},r),(\boldsymbol{\xi}_{k-1},\theta_{k-1})\big) = \langle\boldsymbol{\chi},\boldsymbol{w}\rangle \quad \forall(\boldsymbol{w},r)\in V_{k-1}\times Q_{k-1}. \tag{5.18}$$

It follows from (5.17), (5.16), (4.19), (5.17)–(5.18) and (2.8) that

$$\begin{aligned}\langle\boldsymbol{\chi},\boldsymbol{\zeta}\rangle &= \mathcal{B}\big((\boldsymbol{\zeta},\mu),(\boldsymbol{\xi},\theta)\big)\\ &= \mathcal{B}\big((Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q),(\boldsymbol{\xi},\theta)\big)\\ &= \mathcal{B}\big((Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q),(\boldsymbol{\xi},\theta)-(\boldsymbol{\xi}_{k-1},\theta_{k-1})\big)\\ &= \mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{\xi},\theta)-(\boldsymbol{\xi}_{k-1},\theta_{k-1})\big)\\ &\lesssim (\|\boldsymbol{\xi}-\boldsymbol{\xi}_{k-1}\|_{H^1(\Omega)} + \|\theta-\theta_{k-1}\|_{L_2(\Omega)})\|(\boldsymbol{v},q)\|_{1,k},\end{aligned}$$

which together with (2.11) implies

$$\langle\boldsymbol{\chi},\boldsymbol{\zeta}\rangle \lesssim h_k^{\alpha}\|\boldsymbol{\chi}\|_{H^{-1+\alpha}(\Omega)}\|(\boldsymbol{v},q)\|_{1,k} \quad \forall\,\boldsymbol{\chi}\in[H^{-1+\alpha}(\Omega)]^d. \tag{5.19}$$

The estimate for $\boldsymbol{\zeta}$ follows from (5.19) and duality.

Similarly, we can prove (5.15) by using (2.9), (2.10), (4.20), (5.7) and (5.11). □

5.3. Convergence of the W-cycle algorithm

The following estimates are direct consequences of (5.12)–(5.15):

$$\|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v},q)\|_{1,k} \lesssim m^{-\alpha/2}\|(\boldsymbol{v},q)\|_{1,k} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k, \tag{5.20}$$

$$\|\tilde{R}_k^m(Id_k - I_{k-1}^k \tilde{P}_k^{k-1})(\boldsymbol{v},q)\|_{1,k}^{\sim} \lesssim m^{-\alpha/2}\|(\boldsymbol{v},q)\|_{1,k}^{\sim} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k. \tag{5.21}$$

We can then deduce, by (4.13), (4.21), (5.3), (5.5), (5.7), (5.8) and (5.21),

$$\begin{aligned}
\|(Id_k - I_{k-1}^k P_k^{k-1})S_k^m(\boldsymbol{v},q)\|_{1,k} &\approx \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((Id_k - I_{k-1}^k P_k^{k-1})S_k^m(\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{\|(\boldsymbol{w},r)\|_{1,k}}\\
&= \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v},q),\tilde{R}_k^m(Id_k - I_{k-1}^k \tilde{P}_k^{k-1})(\boldsymbol{w},r)\big)}{\|(\boldsymbol{w},r)\|_{1,k}}\\
&\lesssim \sup_{(\mathbf{0},0)\neq(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\|(\boldsymbol{v},q)\|_{1,k}^{\sim}\|\tilde{R}_k^m(Id_k - I_{k-1}^k \tilde{P}_k^{k-1})(\boldsymbol{w},r)\|_{1,k}^{\sim}}{\|(\boldsymbol{w},r)\|_{1,k}}\\
&\lesssim m^{-\alpha/2}\|(\boldsymbol{v},q)\|_{1,k} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k.
\end{aligned} \tag{5.22}$$

The following lemma provides the contraction number estimate for the two-grid algorithm for (3.10), where the coarse grid residual equation is solved exactly.

Lemma 5.4. *There exists a positive constant C_* independent of k such that*

$$\|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})S_k^{m_1}(\boldsymbol{v},q)\|_{1,k} \le \frac{C_*}{m^{\alpha/2}}\|(\boldsymbol{v},q)\|_{1,k} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k. \tag{5.23}$$

Proof. In view of (4.18), we have

$$(Id_k - I_{k-1}^k P_k^{k-1}) = (Id_k - I_{k-1}^k P_k^{k-1})^2$$

and hence

$$\begin{aligned}
\|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})S_k^{m_1}(\boldsymbol{v},q)\|_{1,k} &= \|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1})S_k^{m_1}(\boldsymbol{v},q)\|_{1,k}\\
&\lesssim m^{-\alpha}\|(\boldsymbol{v},q)\|_{1,k} \quad \forall(\boldsymbol{v},q)\in V_k\times Q_k
\end{aligned}$$

by (5.20) and (5.22). □

A standard perturbation argument (cf. [4, Theorem 5.5]) then leads to the following result for the W-cycle algorithm.

Theorem 5.5. *Let E_k be the error propagation operator for the kth level W-cycle algorithm. For any $C_\dagger > C_*$ (the constant in (5.23)), there exists a positive number m_* (independent of k) such that*

$$\|E_k(\boldsymbol{v},q)\|_{1,k} \le C_\dagger\big(\max(1,m_1)\max(1,m_2)\big)^{-\alpha/2}\|(\boldsymbol{v},q)\|_{1,k} \tag{5.24}$$

for all $(\boldsymbol{v},q)\in V_k\times Q_k$ and $k\ge 0$, provided that $\max(1,m_1)\max(1,m_2)\ge m_*$.

Remark 5.6. It follows from (5.3) and Theorem 5.5 that the kth level W-cycle algorithm for (3.10) is a contraction in the energy norm $|\cdot|_{H^1(\Omega)}+\|\cdot\|_{L_2(\Omega)}$ that appears in the variational formulation of the Oseen problem, provided that $m_1+m_2\ge m_\dagger$ for some sufficiently large positive integer $m_\dagger$ that is independent of k, in which case the contraction number is uniformly bounded away from 1 for all k.

Remark 5.7. We can also derive an analogous result for the W-cycle algorithm for the adjoint problem (3.11).

6. Numerical results

Here we report numerical results for (2.1) on a unit square and an L-shaped domain. The operator L_k in (3.15) is the multigrid $V(1,1)$-cycle solve for the Poisson problem, and the wind function is either $\boldsymbol{\varrho}=(1,0)$ or $(1,1)$. The contraction numbers are obtained by computing the norm of the error propagation operator E_k through the power method.

6.1. Unit square $(0,1)^2$

The contraction numbers for the symmetric W-cycle algorithm with different numbers of smoothing steps are presented in Tables 6.1 and 6.2. The results for the symmetric V-cycle algorithm are presented in Table 6.3.

Table 6.1
Contraction numbers of the W-cycle algorithm on the unit square with $\varrho = (1, 0)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.87	0.88	0.88	0.88	0.88	0.88
(2, 2)	0.76	0.77	0.77	0.77	0.77	0.77
(4, 4)	0.63	0.66	0.66	0.66	0.66	0.66
(8, 8)	0.53	0.54	0.54	0.54	0.54	0.54
(16, 16)	0.37	0.39	0.38	0.38	0.38	0.38
(32, 32)	0.20	0.20	0.20	0.20	0.20	0.20

Table 6.2
Contraction numbers of the W-cycle algorithm on the unit square with $\varrho = (1, 1)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.87	0.88	0.88	0.88	0.88	0.88
(2, 2)	0.76	0.77	0.77	0.77	0.77	0.77
(4, 4)	0.63	0.66	0.66	0.66	0.66	0.66
(8, 8)	0.53	0.54	0.54	0.54	0.54	0.54
(16, 16)	0.38	0.38	0.38	0.38	0.38	0.38
(32, 32)	0.20	0.20	0.20	0.20	0.20	0.20

Table 6.3
Contraction numbers of the V-cycle algorithm on the unit square with $\varrho = (1, 0)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.89	0.90	0.90	0.90	0.90	0.90
(2, 2)	0.80	0.81	0.81	0.81	0.81	0.82
(4, 4)	0.68	0.71	0.71	0.71	0.71	0.72
(8, 8)	0.57	0.60	0.60	0.60	0.60	0.60
(16, 16)	0.43	0.45	0.46	0.46	0.46	0.46
(32, 32)	0.26	0.27	0.27	0.27	0.27	0.27

Table 6.4
Contraction numbers of the W-cycle algorithm on the L-shaped domain with $\varrho = (1, 0)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.89	0.89	0.89	0.89	0.89	0.89
(2, 2)	0.80	0.80	0.79	0.79	0.79	0.79
(4, 4)	0.65	0.67	0.68	0.68	0.68	0.68
(8, 8)	0.55	0.57	0.57	0.57	0.57	0.56
(16, 16)	0.41	0.41	0.41	0.41	0.41	0.41
(32, 32)	0.23	0.23	0.23	0.23	0.23	0.23

The symmetric W-cycle algorithm with one pre-smoothing step and one post-smoothing step appears to be uniformly convergent. The asymptotic decay rate of $1/m$ for the contraction numbers predicted by Theorem 5.5 (with $m_1 = m_2 = m$) is observed in Tables 6.1 and 6.2. The symmetric V-cycle algorithm with one pre-smoothing step and one post-smoothing step also appears to be uniformly convergent, and the asymptotic decay rate for the contract numbers seems to be approaching $1/m$.

6.2. *L-shaped Domain* $(0, 1)^2 \setminus [0.5, 1]^2$

The contraction numbers for the symmetric W-cycle algorithm with different numbers of smoothing steps are presented in Tables 6.4 and 6.5. The results for the symmetric V-cycle algorithm are provided in Table 6.6.

Again both W-cycle and V-cycle algorithms appear to be uniformly convergent with one pre-smoothing and one post-smoothing step. The asymptotic decay rate for the W-cycle algorithm is worse than $1/m$, which is consistent with Theorem 5.5 since the index of elliptic regularity α is strictly less than 1 for the L-shaped domain.

Table 6.5
Contraction numbers of the W-cycle algorithm on the L-shaped domain with $\varrho = (1, 1)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.89	0.89	0.89	0.89	0.89	0.89
(2, 2)	0.80	0.80	0.80	0.79	0.79	0.79
(4, 4)	0.65	0.67	0.68	0.68	0.68	0.68
(8, 8)	0.55	0.56	0.56	0.57	0.57	0.56
(16, 16)	0.40	0.41	0.41	0.41	0.41	0.41
(32, 32)	0.23	0.23	0.23	0.23	0.23	0.23

Table 6.6
Contraction numbers of the V-cycle algorithm on the L-shaped domain $\varrho = (1, 0)$.

(m_1, m_2)	k					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.90	0.91	0.91	0.91	0.91	0.91
(2, 2)	0.84	0.83	0.83	0.83	0.83	0.83
(4, 4)	0.71	0.72	0.73	0.74	0.74	0.74
(8, 8)	0.59	0.62	0.62	0.62	0.62	0.62
(16, 16)	0.46	0.48	0.48	0.48	0.48	0.48
(32, 32)	0.29	0.30	0.30	0.30	0.30	0.30

7. Concluding remarks

We have extended the multigrid algorithms in [4] to the Oseen system and established the uniform convergence of the W-cycle algorithm for a sufficiently large number of smoothing steps. Numerical results indicate that the V-cycle algorithm is also likely to be uniformly convergent if the number of smoothing steps is sufficiently large. Note that the post-smoothing step defined by (4.2) is a Richardson relaxation for a nonstandard second order SPD problem (cf. Remark 4.1). Therefore it may be possible to analyze the V-cycle algorithm in this paper by the additive multigrid theory in [38,39] that is designed for nonstandard problems.

As mentioned in Remark 3.3, the Oseen system can also be solved by the preconditioned GMRES method (cf. [32,33]). We have implemented such a solver with the same block diagonal preconditioner $\mathbb{S}_k$, and compared its performance (solution time and memory requirement) with our multigrid method. For a tolerance that is not very small, the preconditioned GMRES is much faster. However, for a very small tolerance, the solution times for both methods are similar. On the other hand the memory requirement for the multigrid method is always smaller, and it is much smaller than the requirement for the preconditioned GMRES when the tolerance is very small. Therefore our multigrid methods have some advantage over the preconditioned GMRES approach when the tolerance is very small. This advantage may even hold for larger tolerances if the intergrid transfer operators are optimized.

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