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Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities

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SUMMARY

This paper analyzes a multigrid (MG) V-cycle scheme for solving the discretized 2D Poisson equation with corner singularities. Using weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ and a space decomposition based on elliptic projections, we prove that the MG V-cycle with standard smoothers (Richardson, weighted Jacobi, Gauss–Seidel, etc.) and piecewise linear interpolation converges uniformly for the linear systems obtained by finite element discretization of the Poisson equation on graded meshes. In addition, we provide numerical experiments to demonstrate the optimality of the proposed approach. Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: multigrid methods; graded meshes; uniform convergence; corner-like singularities

1. INTRODUCTION

Multigrid (MG) methods are arguably one of the most efficient techniques for solving the large systems of algebraic equations resulting from finite element discretizations of elliptic boundary value problems. Many of the known results on the convergence properties of MG methods for elliptic equations can be found in monographs and survey papers by Bramble [1], Hackbusch [2], Trottenberg *et al.* [3], Xu [4] and the references therein.

It is well known that the geometry of the boundary and changes in the boundary condition can influence the regularity of the solution [5–12]. In particular, if the domain possesses re-entrant corners, cracks, or there exist abrupt changes in the boundary conditions, then the solution of

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the elliptic boundary value problem may have singularities in H^2—we hereafter refer to singularities of these types as corner-like singularities. One possible approach for obtaining accurate numerical approximations to the solutions nearby these types of singularities is to make use of graded meshes [6, 13–15], for which the quasi-optimal convergence rates of the numerical solutions can be recovered by using an analysis based on weighted Sobolev spaces. The analysis of the convergence rate of MG methods in such settings is, however, non-trivial. The difficulties that arise are due primarily to the lack of regularity of the solution and the non-uniformity of the mesh.

A result for the uniform convergence of the MG method assuming full regularity was derived by Braess and Hackbusch [16]; in Brenner's paper [17], the analysis of the convergence rate for only *partial* regularity was presented; Bramble *et al.* [18] developed the convergence estimate without regularity assumptions for an L^2-projection-based decomposition. In addition, on graded meshes, using the approximation property in [14], Yserentant [19] proved the uniform convergence of the MG W-cycle with a particular iterative method on each level for piecewise linear functions. There are also many other more classical convergence proofs that use algebraic techniques and derive convergence results based on assumptions related to, but nevertheless different from, the regularity of the underlying partial differential equation [20, 21].

In this paper, using a space decomposition for elliptic projections and an estimate on the weighted Sobolev space K_a^m, we prove the uniform convergence of the MG V-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss–Seidel, etc.) for elliptic problems with corner-like singularities, discretized using graded meshes. To date, this type of convergence analysis has been carried out only for problems with full elliptic regularity. The result presented here establishes the uniform convergence of the MG method for problems with less regular solutions discretized using graded meshes that appropriately capture the correct behavior of the solution near the singularities. Although the main convergence theorem can be modified for elliptic problems discretized on general graded meshes, for exposition, we restrict our discussion to the graded mesh refinement (GMR) strategy developed by Băcuţă *et al.* [6]. Before proceeding, we mention that, with appropriate modifications, our analysis for linear elements can also be applied to higher-order finite element methods.

1.1. Preliminaries and notation

Let Ω be a bounded polygonal domain, possibly with cracks, in $\mathbb{R}^2$ and consider the following prototype elliptic equation with mixed boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial_D\Omega \\ \partial u/\partial n &= 0 \quad \text{on } \partial_N\Omega \end{aligned} \tag{1}$$

where $\partial_D\Omega$ and $\partial_N\Omega$ consist of segments of the boundary, and we assume that the Neumann boundary condition is not imposed on adjacent sides of the boundary. We note that, in the Sobolev space H^m, corner-like singularities appear in the solution near vertices of the domain. Here, by vertices, we mean the points on $\bar{\Omega}$ where corner-like singularities in $H^2(\Omega)$ are located, namely, the geometric vertices on re-entrant corners, crack points, or points with an interior angle $\theta > \pi/2$, where the boundary conditions change.

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Let $H_D^1(\Omega)=\{u\in H^1(\Omega)|\ u=0 \text{ on } \partial_D\Omega\}$ be the space of $H^1(\Omega)$ functions with zero trace on $\partial_D\Omega$, $\mathscr{T}_j$, $0\leqslant j\leqslant J$, be a sequence of appropriately graded and nested triangulations of Ω, and $\mathscr{M}_j$, $0\leqslant j\leqslant J$, be the finite element space associated with the linear Lagrange triangle [22] on $\mathscr{T}_j$. Then,

$$\mathscr{M}_0\subset\mathscr{M}_1\subset\cdots\subset\mathscr{M}_j\subset\cdots\subset\mathscr{M}_J\subset H_D^1(\Omega)$$

Let A be the differential operator associated with Equation (1). Solving (1) amounts to finding an approximation $u_J\in\mathscr{M}_J$ such that

$$a(u_J,v_J)=(Au_J,v_J)=(\nabla u_J,\nabla v_J)=(f,v_J)\quad \forall v_J\in\mathscr{M}_J$$

Denoting by N_J the dimension of the space $\mathscr{M}_J$, by using a GMR strategy, one can recover the following quasi-optimal rate of convergence for the finite element approximation $u_J\in\mathscr{M}_J$ on $\mathscr{T}_J$:

$$\|u-u_J\|_{H^1(\Omega)}\leqslant CN_J^{-1/2}\|f\|_{L^2(\Omega)}$$

The main objective of this paper is to prove the uniform convergence of the MG V-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss–Seidel, etc.) and linear interpolation applied to the 2D Poisson equation discretized using piecewise linear functions on graded meshes obtained via the GMR strategy introduced in [6]. Moreover, we shall show that the convergence rate, c, of the MG V-cycle satisfies

$$c\leqslant\frac{c_1}{c_1+c_2n}$$

where c_1 and c_2 are mesh-independent constants related to the elliptic equation and the smoother, respectively, and n is the number of iterative solves on each subspace. We note that this result can also be used to estimate the efficiency of other subspace smoothers on graded meshes.

The rest of this paper is organized as follows. In Section 2, we introduce the weighted Sobolev space $K_a^m(\Omega)$ for boundary value problem (1) and review the method of subspace corrections (MSC). In addition, we briefly describe the GMR strategy under consideration here for generating the sequence of graded meshes. Then, in Section 3, we prove the approximation and smoothing properties, which in turn lead to our main MG convergence theorem. Section 4 contains numerical results of the proposed method applied to problem (1).

2. WEIGHTED SOBOLEV SPACES AND THE MSC

In this section, we begin by introducing the weighted Sobolev space $K_a^m(\Omega)$ and the mesh refinement strategy under consideration for recovering quasi-optimal rates of convergence of the finite element solution. Then, we present the MSC and a technique for estimating the norm of the product of non-expansive operators.

2.1. *Weighted Sobolev spaces and graded meshes*

It has been shown in [6–8, 14, 23] that with a careful choice of the parameters in the weight, the singular behavior of the solution in Equation (1) can be captured well in the following weighted Sobolev spaces. Namely, there is no loss of regularity of the solution in these spaces and the corresponding refinements of meshes are optimal in the sense of Theorem 2.3.

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Let $(x, y)\in\bar{\Omega}$ be an arbitrary point and $S=\{S_i\}$ be the set of vertices of the domain, on which the solution has singularities in $H^2(\Omega)$. Denote by $r_i(x, y)$ the distance from (x, y) to the vertex $S_i\in S$ and let $\rho(x, y)$ be a smooth function on $\bar{\Omega}$, such that $\rho=r_i$ in the neighborhood of S_i, and $\rho\geqslant C>0$ otherwise. Then, the weighted Sobolev space $K_a^m(\Omega)$, $m\geqslant 0$, is defined as follows [6, 11]:

$$K_a^m(\Omega)=\{u\in H_{\mathrm{loc}}^m(\Omega)|\ \rho^{i+j-a}\partial_x^i\partial_y^j u\in L^2(\Omega), i+j\leqslant m\}$$

The corresponding K_a^m-norm and seminorm for any function $v\in K_a^m(\Omega)$ are

$$\|v\|_{K_a^m(\Omega)}^2 := \sum_{i+j\leqslant m} \|\rho^{i+j-a}\partial_x^i\partial_y^j v\|_{L^2(\Omega)}^2$$

$$|v|_{K_a^m(\Omega)}^2 := \sum_{i+j=m} \|\rho^{m-a}\partial_x^i\partial_y^j v\|_{L^2(\Omega)}^2$$

Note that ρ is equal to the distance function $r_i(x, y)$ near the vertex S_i. Thus, we have the following proposition and mesh refinements as in [6, 15].

Proposition 2.1
We have $|v|_{K_1^1(\Omega)}\eqsim|v|_{H^1(\Omega)}$, $\|v\|_{K_1^0(\Omega)}\geqslant C\|v\|_{L^2(\Omega)}$, and the Poincaré type inequality $\|v\|_{K_1^0(\Omega)}\leqslant C|v|_{K_1^1(\Omega)}$ for $v\in K_1^1(\Omega)\cap\{v|_{\partial_D\Omega}=0\}$.

Here, $a\eqsim b$ means there exist positive constants C_1, C_2, such that $C_1 b\leqslant a\leqslant C_2 b$.

Definition 2.2
Let κ be the ratio of decay of triangles near a vertex $S_i\in S$. Then, for every $\varepsilon<\min(\pi/t\alpha_i)$, one can choose $\kappa=2^{-1/\varepsilon}$, where α_i is the interior angle of vertex S_i, $t=1$ on vertices with both Dirichlet boundary conditions, and $t=2$ if the boundary condition changes type at S_i. For example, $\alpha_i=2\pi$ and $t=1$ on crack points with both Dirichlet boundary conditions. In the initial triangulation, we require that each triangle contains at most one point in S, and each S_i needs to be a vertex of some triangle. In other words, no point in S is sitting on the edge or in the interior of a triangle. Let $\mathscr{T}_j=\{T_k\}$ be the triangulation after j refinements. Then, for the $(j+1)$th refinement, if the function ρ is bounded away from 0 on a triangle (no point in S contained), new triangles are obtained by connecting the mid-points of the old one. However, if S_i is one of the vertices of a triangle $\triangle S_i BC$, then we choose a point D on $\overline{S_i B}$ and another point E on $\overline{S_i C}$ such that the following holds for the ratios of the lengths

$$\kappa=\overline{S_i D}/\overline{S_i B}=\overline{S_i E}/\overline{S_i C}$$

In this way, the triangle $\triangle S_i BC$ is divided into four smaller triangles by connecting D, E, and the mid-point of $\overline{BC}$ (see Figure 1).

We note that other refinements, for example, those found in [13, 14] also satisfy this condition, although they follow different constructions. We now conclude this subsection by restating the following theorem derived in [6, 15].

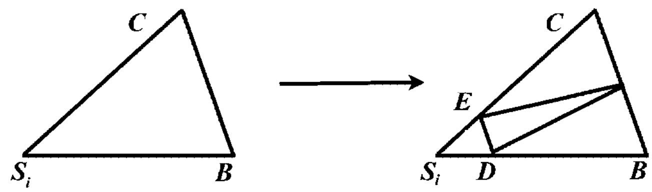

Figure 1. Mesh refinements: triangulation after one refinement, $\kappa=0.2$.

Theorem 2.3
Let $u_j \in \mathscr{M}_j$ be the finite element solution of Equation (1) and denote by N_j the dimension of $\mathscr{M}_j$. Then, there exists a constant $B_1=B_1(\Omega,\kappa,\varepsilon)$, such that

$$\|u-u_j\|_{H^1(\Omega)} \leqslant B_1 N_j^{-1/2}\|f\|_{K^0_{\varepsilon-1}(\Omega)} \leqslant B_1 N_j^{-1/2}\|f\|_{L^2(\Omega)}$$

for every $f\in L^2(\Omega)$, where $\varepsilon<1$ is determined from Definition 2.2, $\mathscr{M}_j$ is the finite element space of linear functions on the graded mesh $\mathscr{T}_j$, as described in the introduction.

Remark 2.4
For $u\notin H^2(\Omega)$, this theorem follows from the fact that the differential operator $A: K^{m+1}_{1+\varepsilon}(\Omega)\cap\{u=0,\text{ on } \partial_D\Omega\}\rightarrow K^{m-1}_{-1+\varepsilon}(\Omega), m\geqslant 0$, in Equation (1), is an isomorphism between the weighted Sobolev spaces.

2.2. *The method of subspace corrections*

In this subsection, we review the MSC and provide an identity for estimating the norm of the product of non-expansive operators. In addition, Lemma 2.6 reveals the connection between the matrix representation and operator representation of the MG method.

Let $H^1_D(\Omega)=\{u\in H^1(\Omega)|u=0 \text{ on } \partial_D\Omega\}$ be the Hilbert space associated with Equation (1), $\mathscr{T}_j$ be the associated graded mesh, as defined in the previous subsection, $\mathscr{M}_j\in H^1_D(\Omega)$ be the space of piecewise linear functions on $\mathscr{T}_j$, and $A: H^1_D(\Omega)\rightarrow (H^1_D(\Omega))'$ be the corresponding differential operator. The weak form for (1) is then

$$a(u,v)=(Au,v)=(-\Delta u,v)=(\nabla u,\nabla v)=(f,v)\quad \forall v\in H^1_D(\Omega)$$

where the pairing $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. Here, $a(\cdot,\cdot)$ is a continuous bilinear form on $H^1_D(\Omega)\times H^1_D(\Omega)$ and by the Poincare inequality is also coercive. In addition, since the $\mathscr{T}_j$ are nested,

$$\mathscr{M}_0\subset\mathscr{M}_1\subset\cdots\subset\mathscr{M}_j\subset\cdots\subset\mathscr{M}_J\subset H^1_D(\Omega)$$

Define $Q_j,\ P_j: H^1_D(\Omega)\rightarrow\mathscr{M}_j$ and $A_j:\mathscr{M}_j\rightarrow\mathscr{M}_j$ as orthogonal projectors and the restriction of A on $\mathscr{M}_j$, respectively,

$$(Q_j u, v_j)=(u,v_j),\quad a(P_j u,v_j)=a(u,v_j)$$

$$(Au_j,v_j)=(A_j u_j,v_j)\quad \forall u\in H^1_D(\Omega)\ \ \forall u_j,v_j\in\mathscr{M}_j$$

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Let $\mathcal{N}_j = \{x_i^j\}$ be the set of nodal points in $\mathcal{T}_j$ and $\phi_k(x_i^j) = \delta_{i,k}$ be the linear finite element nodal basis function corresponding to node x_k^j. Then, the jth level finite element discretization reads: Find $u_j \in \mathcal{M}_j$, such that

$$A_j u_j = f_j \tag{2}$$

where $f_j \in \mathcal{M}_j$ satisfies $(f_j, v_j) = (f, v_j)$, $\forall v_j \in \mathcal{M}_j$.

The MSC reduces an MG process to choosing a sequence of subspaces and corresponding operators $B_j : \mathcal{M}_j \to \mathcal{M}_j$ approximating A_j^{-1}, $j = 1, \ldots, J$. For example, in the MSC framework, the standard MG backslash cycle for solving (2) is defined by the following subspace correction scheme:

$$u_{j,l} = u_{j,l-1} + B_j(f_j - A_j u_{j,l-1})$$

where the operators $B_j : \mathcal{M}_j \to \mathcal{M}_j$, $0 \leqslant j \leqslant J$, are recursively defined as follows [24].

Algorithm 2.5
Let $R_j \approx A_j^{-1}$, $j > 0$, denote a local relaxation method. For $j = 0$, define $B_0 = A_0^{-1}$. Assume that $B_{j-1} : \mathcal{M}_{j-1} \to \mathcal{M}_{j-1}$ is defined. Then,

1. Fine grid smoothing: For $u_j^0 = 0$ and $k = 1, 2, \ldots, n$,

$$u_j^k = u_j^{k-1} + R_j(f_j - A_j u_j^{k-1}) \tag{3}$$

2. Coarse grid correction: Find the corrector $e_{j-1} \in \mathcal{M}_{j-1}$ by the iterator B_{j-1}

$$e_{j-1} = B_{j-1} Q_{j-1}(f_j - A_j u_j^n)$$

Then, $B_j f_j = u_j^n + e_{j-1}$.

Recursive application of Algorithm 2.5 results in an MG V-cycle for which the following identity holds: $I - B_J^v A_J = (I - B_J A_J)^*(I - B_J A_J)$ [24], where B_J^v is the iterator for the MG V-cycle. Direct computation gives the following useful result:

$$\begin{aligned} u_j^n &= (I - R_j A_j) u_j^{n-1} + R_j A_j u_j \\ &= (I - R_j A_j)^2 u_j^{n-2} - (I - R_j A_j)^2 u_j + u_j \\ &= -(I - R_j A_j)^n u_j + u_j \end{aligned}$$

where u_j is the finite element solution of (2) and u_j^n is the approximation after n iterations of (3) on the jth level. Let $T_j = (I - (I - R_j A_j)^n) P_j$ be a linear operator and define $T_0 = P_0$. We have the following identity:

$$\begin{aligned} (I - B_J A_J) u_J = u_J - u_J^n - e_{J-1} &= (I - T_J) u_J - e_{J-1} \\ &= (I - B_{J-1} A_{J-1} P_{J-1})(I - T_J) u_J \end{aligned}$$

where, for $B_{J-1} = A_{J-1}^{-1}$, this becomes a two-level method. Recursive application of this identity then yields the error propagation operator of an MG V-cycle:

$$(I - B_J A_J) = (I - T_0)(I - T_1) \cdots (I - T_J)$$

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To estimate the uniform convergence of the MG V-cycle, we thus need to show that

$$\|I - B_J^v A_J\|_a = \|I - B_J A_J\|_a^2 \leqslant c < 1$$

where c is independent of J and $\|u\|_a^2 = a(u,u) = (Au, u)$ on Ω.

Associated with each T_j, we introduce its symmetrization

$$\bar{T}_j = T_j + T_j^* - T_j^* T_j$$

where T_j^* is the adjoint operator of T_j with respect to the inner product $a(\cdot,\cdot)$. By a well-known result found in [25], the following estimate holds:

$$\|I - B_J A_J\|_a^2 = \frac{c_0}{1+c_0}$$

where

$$c_0 \leqslant \sup_{\|v\|_a=1} \sum_{j=1}^{J} a((\bar{T}_j^{-1} - I)(P_j - P_{j-1})v, (P_j - P_{j-1})v) \tag{4}$$

Now, to prove the uniform convergence of the proposed MG scheme, we must derive a uniform bound on the constant c_0.

Although the above presentation is in terms of operators, the matrix representation of the smoothing step (3) is often used in practice. By the matrix representation $\mathbf{R}$ of an operator R on $\mathscr{M}_j$, we here mean that with respect to the basis $\{\phi_i\}_{i=1}^{N_j}$ of $\mathscr{M}_j$,

$$R(\phi_k) = \sum_{i=1}^{N_j} \mathbf{R}_{i,k} \phi_i$$

where $\mathbf{R}_{i,k}$ is the (i,k) component of the matrix $\mathbf{R}$. Throughout the paper, we use boldfaced letters to denote vectors and matrices.

Let $\mathbf{A}_S = \mathbf{D} - \mathbf{L} - \mathbf{U}$ be the stiffness matrix associated with the operator A_j, where the matrix $\mathbf{D}$ consists of only the diagonal entries of $\mathbf{A}_S$, while matrices $-\mathbf{L}$ and $-\mathbf{U}$ are the strictly lower and upper triangular parts of $\mathbf{A}_S$, respectively. Denote by $\mathbf{R}_M$ the corresponding matrix of the smoother R_j on the jth level. For example, $\mathbf{R}_M = \mathbf{D}^{-1}$ for the Jacobi method, and $\mathbf{R}_M = (\mathbf{D} - \mathbf{L})^{-1}$ for the Gauss–Seidel method. In addition, let $\mathbf{u}^l$, $\mathbf{u}^{l-1}$, and $\mathbf{f}$ be the vectors containing the coordinates of u_j^l, u_j^{l-1}, $f_j \in \mathscr{M}_j$ on the basis $\{\phi_i\}_{i=1}^{N_i}$, namely $u_j^l = \sum_{i=1}^{N_j} \mathbf{u}_i^l \phi_i$. Then, one smoothing step for solving (2) on a single level j in terms of matrices reads

$$\mathbf{u}^l = \mathbf{u}^{l-1} + \mathbf{R}_M(\mathbf{Mf} - \mathbf{A}_S \mathbf{u}^{l-1}) \tag{5}$$

where $\mathbf{M}$ is the mass matrix, and $\mathbf{M}_{i,k} = (\phi_i, \phi_k)$.

Lemma 2.6

Let $\mathbf{R}$ be the matrix representation of the smoother R_j in Equation (3). Then,

$$\mathbf{R} = \mathbf{R}_M \mathbf{M}$$

Hence,

$$R_j(\phi_k) = \sum_{i=1}^{N_j} \mathbf{R}_{i,k} \phi_i = \sum_{i=1}^{N_j} (\mathbf{R}_M \mathbf{M})_{i,k} \phi_i$$

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and

$$\mathbf{u}^l = \mathbf{u}^{l-1} + \mathbf{R}_M(\mathbf{Mf} - \mathbf{A}_S\mathbf{u}^{l-1}) = \mathbf{u}^{l-1} + \mathbf{R}(\mathbf{f} - \mathbf{M}^{-1}\mathbf{A}_S\mathbf{u}^{l-1})$$

Proof
Denote by $\mathbf{A}$ the matrix representation of the operator A. Note that

$$(A\phi_i, \phi_k) = \left(\sum_{m=1}^{N_j} \mathbf{A}_{m,i}\phi_m, \phi_k\right) = (\nabla\phi_k, \nabla\phi_i) = (\mathbf{A}_S)_{k,i}$$

indicates $\mathbf{A}_S = \mathbf{MA}$. Moreover, in terms of matrices and vectors, Equation (3) also reads

$$\sum_{i=1}^{N_j} \mathbf{u}_i^l \phi_i = \sum_{i=1}^{N_j} \mathbf{u}_i^{l-1}\phi_i + \sum_{i=1}^{N_j}\sum_{k=1}^{N_j} \mathbf{R}_{k,i}\mathbf{f}_i\phi_k - \sum_{i=1}^{N_j}\sum_{k=1}^{N_j}\sum_{m=1}^{N_j} \mathbf{R}_{m,k}\mathbf{A}_{k,i}\mathbf{u}_i\phi_m$$

Then, the inner product with ϕ_n on both sides, $1 \leqslant n \leqslant N_j$, leads to

$$\mathbf{Mu}^l = \mathbf{Mu}^{l-1} + \mathbf{MRf} - \mathbf{MRAu}$$

Multiplication by $\mathbf{M}^{-1}$ gives

$$\mathbf{u}^l = \mathbf{u}^{l-1} + \mathbf{R}(\mathbf{f} - \mathbf{Au})$$

Taking into account that Equations (3) and (5) represent the same iteration, we have

$$\mathbf{Rf} = \mathbf{R}_M\mathbf{Mf}$$

Note the above equation holds for any $\mathbf{f} \in \mathbb{R}^{N_j}$. Therefore, $\mathbf{R} = \mathbf{R}_M\mathbf{M}$, which completes the proof. $\square$

3. UNIFORM CONVERGENCE OF THE MG METHOD ON GRADED MESHES

Next, we derive an estimate for the constant c_0 in (4) of Section 2 and then proceed to establish the main convergence theorem of the paper. We begin by proving several lemmas that are needed in the convergence proof. For simplicity, we assume that there is only a single point $S_0 \in \bar{\Omega}$, for which the solution of Equation (1) has a singularity in $H^2(\Omega)$, and that a nested sequence of graded meshes has been constructed, as described in Definition 2.2. The same argument, however, carries over to problems on domains with multiple singularities and also for similar refinement strategies.

Denote by $\{T_i^{S_0}\}$ all the initial triangles with the common vertex S_0. Recall that the function ρ in the weight equals the distance to S_0 on these triangles. Based on the process in Definition 2.2, after N refinements, the region $\cup T_i^{S_0}$ is partitioned into $N+1$ sub-domains (*layers*) D_n, $0 \leqslant n \leqslant N$, whose sizes decrease by the factor κ as they approach S_0 (see Figure 2). In addition, $\rho(x, y) \eqsim \kappa^n$ on D_n for $0 \leqslant n < N$ and $\rho(x, y) \leqslant C\kappa^N$ on D_N. Meanwhile, sub-triangles (nested meshes) are generated in these *layers* D_n, $0 \leqslant n \leqslant N$, with corresponding mesh size of order $O(\kappa^n 2^{n-N})$.

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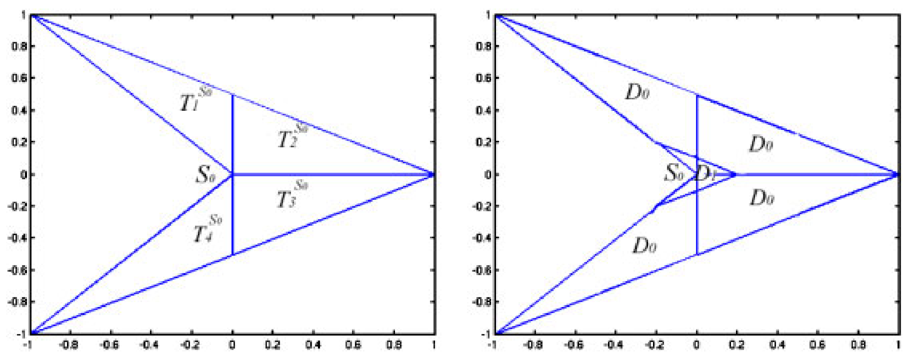

Figure 2. Initial triangles with vertex S_0 (left); layer D_0 and D_1 after one refinement (right), $\kappa=0.2$.

Note that $\Omega=(\cup D_n)\cup(\Omega\backslash\cup D_n)$. Let ∂D_n be the boundary of D_n. Then, we define a piecewise constant function $r_p(x,y)$ on $\bar{\Omega}$ as follows.

$$r_p(x,y)=\begin{cases}(1/2\kappa)^n & \text{on } \bar{D}_n\backslash\partial D_{n-1} \quad \text{for } 1<n\leqslant N\\ 1 & \text{otherwise}\end{cases}$$

where $N=J$ is the number of refinements for $\mathscr{T}_J$. Therefore, the restriction of r_p on every $T_i^{S_0}\cap D_n$ is a constant. Recall that $\varepsilon<1$ is the parameter for κ, such that $\kappa=2^{-1/\varepsilon}$. Define the weighted inner product with respect to r_p:

$$(u,v)_{r_p}=(r_p u, r_p v)=\int_\Omega r_p^2 uv$$

In addition, the above inner product induces the norm:

$$\|u\|_{r_p}=(u,u)_{r_p}^{1/2}$$

Then, the following estimate holds.

Lemma 3.1

$$(u_j-P_{j-1}u_j, u_j-P_{j-1}u_j)_{r_p}\leqslant\frac{c_1}{N_j}a(u_j-P_{j-1}u_j, u_j-P_{j-1}u_j) \quad \forall u_j\in\mathscr{M}_j$$

where $N_j=O(2^{2j})$ is the dimension of $\mathscr{M}_j$.

Proof

This lemma can be proved by the duality argument as follows.

Consider the following boundary value problem:

$$-\Delta w=r_p^2(u_j-P_{j-1}u_j) \quad \text{in } \Omega$$

$$w=0 \quad \text{on } \partial_D\Omega$$

$$\partial w/\partial n=0 \quad \text{on } \partial_N\Omega$$

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Then, since $P_{j-1}w \in \mathscr{M}_{j-1}$, from the equation above, we have

$$\begin{aligned}(r_p(u_j - P_{j-1}u_j), r_p(u_j - P_{j-1}u_j)) &= (r_p^2(u_j - P_{j-1}u_j), u_j - P_{j-1}u_j)\\ &= (\nabla w, \nabla(u_j - P_{j-1}u_j))\\ &= (\nabla(w - P_{j-1}w), \nabla(u_j - P_{j-1}u_j))\end{aligned}$$

We note that Δw is a piecewise linear function on the graded triangulation $\mathscr{T}_j$ that is derived after j refinements. From the results of Theorem 2.3, we conclude

$$\begin{aligned}|w - P_{j-1}w|^2_{H^1(\Omega)} &\leqslant (C_1/N_{j-1})\|\Delta w\|^2_{K^0_{\varepsilon-1}(\Omega)}\\ &= (C_1/N_{j-1})\left(\sum_{n=0}^{j} \|\rho^{1-\varepsilon}\Delta w\|^2_{L^2(D_n)} + \|\rho^{1-\varepsilon}\Delta w\|^2_{L^2(\Omega\backslash\cup D_n)}\right)\\ &\leqslant (C/N_{j-1})\left(\sum_{n=0}^{j} \|\kappa^{n(1-\varepsilon)}\Delta w\|^2_{L^2(D_n)} + \|\Delta w\|^2_{L^2(\Omega\backslash\cup D_n)}\right)\\ &= (C/N_{j-1})\left(\sum_{n=0}^{j} \|2^n\kappa^n\Delta w\|^2_{L^2(D_n)} + \|\Delta w\|^2_{L^2(\Omega\backslash\cup D_n)}\right)\\ &= (C/N_{j-1})\left(\sum_{n=0}^{j} \|r_p^{-1}\Delta w\|^2_{L^2(D_n)} + \|\Delta w\|^2_{L^2(\Omega\backslash\cup D_n)}\right)\\ &= (C/N_{j-1})\|r_p^{-1}\Delta w\|^2_{L^2(\Omega)}\end{aligned}$$

The inequalities above are based on the definition of κ, r_p, and related norms. Now, since $N_j = O(N_{j-1})$, combining the results above, we have

$$\begin{aligned}\|u_j - P_{j-1}u_j\|^2_{r_p} &\leqslant \frac{|w - P_{j-1}w|^2_{H^1}|u_j - P_{j-1}u_j|^2_{H^1}}{\|(u_j - P_{j-1}u_j)\|^2_{r_p}}\\ &= \frac{|w - P_{j-1}w|^2_{H^1}|u_j - P_{j-1}u_j|^2_{H^1}}{\|r_p^{-1}\Delta w\|^2_{L^2}}\\ &\leqslant \frac{c_1}{N_j}|u_j - P_{j-1}u_j|^2_{H^1} = \frac{c_1}{N_j}a(u_j - P_{j-1}u_j, u_j - P_{j-1}u_j)\end{aligned}$$

which completes the proof. □

Recall that the matrix form $\mathbf{R}_M$ and the matrix representation $\mathbf{R}$ of a smoother R_j are different from Lemma 2.6. Then, we have the following result regarding the smoother $\bar{R}_j = R_j + R_j^t - R_j^t A_j R_j$ on $\mathscr{M}_j$, which is the symmetrization of R_j, where R_j^t is the adjoint of R_j with respect to $(\cdot, \cdot)$.

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Lemma 3.2

For the subspace smoother $\bar{R}_j : \mathscr{M}_j \to \mathscr{M}_j$, we assume that there is a constant $C>0$ independent of j, such that the corresponding matrix form $\bar{\mathbf{R}}_M$ satisfies

$$\mathbf{v}^{\mathrm{T}}\bar{\mathbf{R}}_M\mathbf{v} \geqslant C\mathbf{v}^{\mathrm{T}}\mathbf{v} \quad \forall \mathbf{v}\in\mathbb{R}^{N_j}$$

on every level j, where N_j is the dimension of the subspace $\mathscr{M}_j$. Then, there exists $c_2>0$, also independent of the level j, such that the following estimate holds on each graded mesh $\mathscr{T}_j$,

$$\frac{c_2}{N_j}(\bar{R}_j v, v) \leqslant (\bar{R}_j v, \bar{R}_j v)_{r_p} \quad \forall v\in\mathscr{M}_j$$

Proof

For any $v=\sum_i \mathbf{v}_i\phi_i \in \mathscr{M}_j$, from Lemma 2.6, we have

$$(\bar{R}_j v, v) = \left(\sum_m \mathbf{v}_m \sum_k (\bar{\mathbf{R}}_M\mathbf{M})_{k,m}\phi_k, \sum_i \mathbf{v}_i\phi_i\right) = \mathbf{v}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\bar{\mathbf{R}}_M\mathbf{M}\mathbf{v}$$

On the other hand,

$$\begin{aligned}(\bar{R}_j v, \bar{R}_j v)_{r_p} &= \left(\sum_m \mathbf{v}_m \sum_k (\bar{\mathbf{R}}_M\mathbf{M})_{k,m}\phi_k, \sum_l \mathbf{v}_l \sum_i (\bar{\mathbf{R}}_M\mathbf{M})_{i,l}\phi_i\right)\\ &= \mathbf{v}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\bar{\mathbf{R}}_M\tilde{\mathbf{M}}\bar{\mathbf{R}}_M\mathbf{M}\mathbf{v}\end{aligned}$$

where $\tilde{\mathbf{M}}$ is a matrix satisfying $(\tilde{\mathbf{M}})_{i,k} = (r_p\phi_i, r_p\phi_k)$. Note that both $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are symmetric positive definite (SPD). Now, suppose $\mathrm{supp}(\phi_i)\cap D_n \neq \emptyset$, $0\leqslant n\leqslant j$. Then, on $\mathrm{supp}(\phi_i)$, the mesh size is $O(\kappa^n 2^{n-j})$ and $r_p \eqsim (1/2\kappa)^n$, respectively, since $\mathrm{supp}(\phi_i)$ is covered by at most two adjacent layers. Thus, all the non-zero elements in $\tilde{\mathbf{M}}$ are positive and $\tilde{\mathbf{M}} \eqsim 2^{-2j} \eqsim 1/N_j$. To complete the proof, it is sufficient to show that there exists $C>0$, such that

$$\mathbf{w}^{\mathrm{T}}\bar{\mathbf{R}}_M^{1/2}\tilde{\mathbf{M}}\bar{\mathbf{R}}_M^{1/2}\mathbf{w} \geqslant (C/N_j)\mathbf{w}^{\mathrm{T}}\mathbf{w}$$

where $\mathbf{w} = \bar{\mathbf{R}}_M^{1/2}\mathbf{M}\mathbf{v}$.

From the condition on $\bar{\mathbf{R}}_M$ and the estimates on $\tilde{\mathbf{M}}$, it follows that

$$\mathbf{w}^{\mathrm{T}}\bar{\mathbf{R}}_M^{1/2}\tilde{\mathbf{M}}\bar{\mathbf{R}}_M^{1/2}\mathbf{w} \eqsim (1/N_j)\mathbf{w}^{\mathrm{T}}\bar{\mathbf{R}}_M\mathbf{w} \geqslant (C/N_j)\mathbf{w}^{\mathrm{T}}\mathbf{w} \qquad \square$$

Remark 3.3

For our choice of graded meshes, the triangles remain shape-regular elements, that is, the minimum angles of the triangles are bounded away from 0. Therefore, the stiffness matrix $\mathbf{A}_S$ has a bounded number of non-zero entries per row and each entry is of order $O(1)$. Hence, the maximum eigenvalue of $\mathbf{A}_S$ is bounded. For this reason, standard smoothers (Richardson, weighted Jacobi, Gauss–Seidel, etc.) satisfy Lemma 3.2, and $(\mathbf{R}_M)_{i,j} = O(1)$ as well, since they are all from part of the matrix $\mathbf{A}_S$. Moreover, if $\mathbf{R}_M$ is SPD and the spectral radius $\rho(\mathbf{R}_M\mathbf{A}_S)\leqslant\omega$, for $0<\omega<1$, then based on Lemma 2.6,

$$\begin{aligned}a(R_jA_jv, v) &= (A_jR_jA_jv, v)\\ &= \mathbf{v}^{\mathrm{T}}\mathbf{A}_S\mathbf{R}_M\mathbf{A}_S\mathbf{v}\\ &\leqslant \omega a(v, v)\end{aligned}$$

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The last inequality follows from the similarity of the matrix $\mathbf{A}_S^{1/2}\mathbf{R}_M\mathbf{A}_S^{1/2}$ and the matrix $\mathbf{R}_M\mathbf{A}_S$. Note that the above inequality implies the spectral radius of $R_jA_j \leqslant \omega$, since R_jA_j is symmetric with respect to $a(\cdot,\cdot)$.

We then define the following operators for the MG V-cycle. Recall T_j from Section 2 and let R_j denote a subspace smoother satisfying Lemma 3.2. Recall the symmetrization $\bar{R}_j$ of R_j, and assume the spectral radius $\rho(\bar{R}_jA_j)\leqslant\omega$ for $0<\omega<1$. Note that R_j^t is the adjoint of R_j with respect to $(\cdot,\cdot)$ and T_j^* is the adjoint of T_j with respect to $a(\cdot,\cdot)$. With n smoothing steps, where R_j and R_j^t are applied alternatingly, the operator G_j and G_j^* are defined as follows:

$$G_j = I - R_jA_j, \quad G_j^* = I - R_j^tA_j$$

With this choice

$$T_j = \begin{cases} P_j - (G_j^*G_j)^{n/2}P_j & \text{for even } n \\ P_j - G_j(G_j^*G_j)^{(n-1)/2}P_j & \text{for odd } n \end{cases}$$

Therefore, if we define

$$G_{j,n} = \begin{cases} G_j^*G_j & \text{for even } n \\ G_jG_j^* & \text{for odd } n \end{cases}$$

since $P_j^2 = P_j$,

$$\bar{T}_j = T_j + T_j^* - T_j^*T_j = (I - G_{j,n}^n)P_j$$

Note that $\bar{T}_j$ is invertible on $\mathscr{M}_j$, and hence $\bar{T}_j^{-1}$ exists.

The main result concerning the uniform convergence of the MG V-cycle for our model problem is summarized in the following theorem.

Theorem 3.4

On every triangulation $\mathscr{T}_j$, suppose that the smoother on each subspace $\mathscr{M}_j$ satisfies Lemma 3.2. Then, following the algorithm described above, we have

$$\|I - B_JA_J\|_a^2 = \frac{c_0}{1+c_0} \leqslant \frac{c_1}{c_1 + c_2 n}$$

where c_1 and c_2 are constants from Lemmas 3.1 and 3.2.

Proof

Recall (4) from Section 2. To estimate the constant c_0, we first consider the decomposition $v = \sum_j v_j$ for any $v \in \mathscr{M}_J$ with

$$v_j = (P_j - P_{j-1})v \in \mathscr{M}_j$$

Then, Lemma 3.1 implies

$$N_j(v_j, v_j)_{r_p} \leqslant c_1 a(v_j, v_j)$$

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Estimating the identity of Xu and Zikatanov [25], we have

$$\begin{aligned} a(\bar{T}_j^{-1}(I-\bar{T}_j)v_j, v_j) &= a((I-G_{j,n}^n)^{-1}G_{j,n}^n v_j, v_j) \\ &= (\bar{R}_j^{-1}\bar{R}_j A_j (I-G_{j,n}^n)^{-1}G_{j,n}^n v_j, v_j) \\ &= (\bar{R}_j^{-1}(I-G_{j,n})(I-G_{j,n}^n)^{-1}G_{j,n}^n v_j, v_j) \end{aligned}$$

Note that $G_{j,n}^k$, $k\leqslant n$, is in fact a polynomial of $\bar{R}_j A_j$. Therefore, $\bar{R}_j^{-1/2}(I-G_{j,n})\bar{R}_j^{1/2}$, $\bar{R}_j^{-1/2}G_{j,n}^n \bar{R}^{1/2}$, and $\bar{R}_j^{-1/2}(I-G_{j,n}^n)\bar{R}_j^{1/2}$ are all polynomials of $\bar{R}_j^{1/2}A_j\bar{R}_j^{1/2}$, where $\bar{R}_j^{-1/2}(I-G_{j,n}^n)\bar{R}_j^{1/2} = (\bar{R}_j^{-1/2}(I-G_{j,n}^n)^{-1}\bar{R}_j^{1/2})^{-1}$. Thus, it can be seen that $\bar{R}_j^{-1/2}(I-G_{j,n})\bar{R}_j^{1/2}$, $\bar{R}_j^{-1/2}G_{j,n}^n\bar{R}^{1/2}$, and $\bar{R}_j^{-1/2}(I-G_{j,n}^n)^{-1}\bar{R}_j^{1/2}$ commute with each other; hence, $\bar{R}_j^{-1/2}(I-G_{j,n})(I-G_{j,n}^n)^{-1}G_{j,n}^n\bar{R}^{1/2}$ is symmetric with respect to $(\cdot,\cdot)$.

Then, based on the above argument, defining $w_j=\bar{R}_j^{-1/2}v_j$, we have

$$\begin{aligned} a(\bar{T}_j^{-1}(I-\bar{T}_j)v_j, v_j) &= (\bar{R}_j^{-1/2}(I-G_{j,n})(I-G_{j,n}^n)^{-1}G_{j,n}^n\bar{R}^{1/2}w_j, w_j) \\ &\leqslant \max_{t\in[0,1]}(1-t)(1-t^n)^{-1}t^n(\bar{R}_j^{-1}v_j, v_j) \\ &\leqslant \frac{1}{n}(\bar{R}_j^{-1}v_j, v_j)\leqslant\frac{N_j}{c_2 n}(v_j, v_j)_{r_p} \end{aligned}$$

where the last inequality is from Lemma 3.2. Moreover,

$$\sum_{j=0}^{J} a(\bar{T}_j^{-1}(I-\bar{T}_j)v_j, v_j)\leqslant\sum_{j=1}^{J}\frac{N_j}{c_2 n}(v_j, v_j)_{r_p}\leqslant\sum_{j=0}^{J}\frac{c_1}{c_2 n}a(v_j, v_j)=\frac{c_1}{c_2 n}a(v, v)$$

Therefore, $c_0\leqslant c_1/(c_2 n)$ and consequently, the MSC yields the following convergence estimate for the MG V-cycle:

$$\|I-B_J A_J\|_a^2=\frac{c_0}{1+c_0}\leqslant\frac{c_1}{c_1+c_2 n}$$

which completes the proof. $\square$

4. NUMERICAL ILLUSTRATION

This section contains numerical results for the proposed MG V-cycle applied to the 2D Poisson equation with a single corner-like singularity. The model test problem we consider here is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6}$$

where the singularity occurs at the tip of the crack $\{(x, y), 0\leqslant x\leqslant 0.5, y=0.5\}$ for $\Omega=(0,1)\times(0,1)$ as in Figure 3.

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The MG scheme used to solve (6) is a standard MG V-cycle with linear interpolation. The sequence of coarse-level problems defining the MG hierarchy is obtained by re-discretizing (6) on the nested meshes constructed using the GMR strategy described in Section 2. The reported results are for $V(1,1)$-cycles and Gauss–Seidel (GS) as a smoother. The asymptotic convergence factors are computed using 100 $V(1,1)$-cycles applied to the homogeneous problem starting with an $O(1)$ random initial approximation.

The asymptotic convergence factors reported in Table I clearly demonstrate our theoretical estimates in that they are independent of the number of refinement levels. To obtain a more complete picture of the overall effectiveness of our MG solver, we examine also storage and work-per-cycle measures. These are usually expressed in terms of *operator complexity*, defined as the number of non-zero entries stored in the operators on all levels divided by the number of non-zero entries in the finest-level matrix, and *grid complexity* defined as the sum of the dimensions of operators over all levels divided by the dimension of the finest-level operator. The grid and, especially, the operator complexities can be viewed as proportionality constants that indicate how expensive the entire V-cycle is compared with performing only the finest-level relaxations of the V-cycle. For our test problem, the grid and operator complexities were 1.2 and 1.3, respectively, independent of the number of levels. Considering the low grid and operator complexities the performance of the resulting MG solver applied to problem (6) is comparable to that of standard geometric MG applied to the Poisson equation with full regularity, i.e. without corner-like singularities; for the Poisson equation discretized on uniformly refined grids, standard MG with a GS smoother and linear interpolation yields $\rho_{\mathrm{MG}} \approx 0.35$.

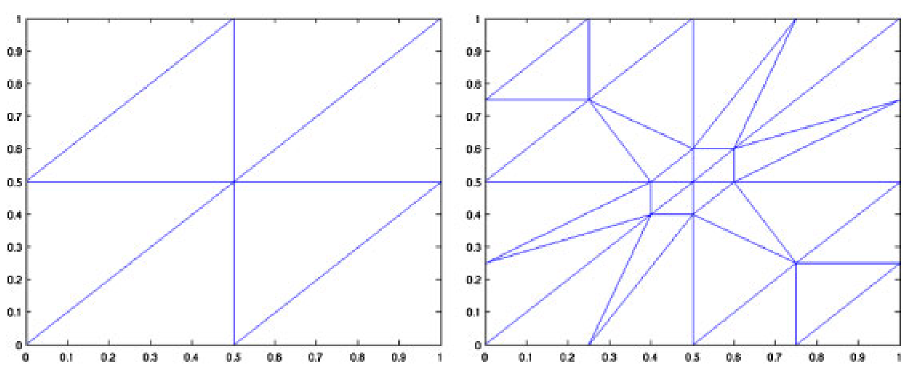

Figure 3. Crack: initial triangulation (left) and the triangulation after one refinement (right), $\kappa = 0.2$.

Table I. Asymptotic convergence factors (ρ_{MG}) for the MG $V(1,1)$-cycle applied to problem (6) with Gauss–Seidel smoother.

levels	2	3	4	5	6
ρ_{MG} (GS)	0.40	0.53	0.56	0.53	0.50

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