SIAM J. NUMER. ANAL.
Vol. 57, No. 5, pp. 2246–2265

INTERIOR ESTIMATES OF FINITE VOLUME ELEMENT METHODS OVER QUADRILATERAL MESHES FOR ELLIPTIC EQUATIONS*

LI GUO†, HENGGUANG LI‡, AND QINGSONG ZOU§

Abstract. In this paper, we study the interior error estimates of a class of finite volume element methods (FVEMs) over quadrilateral meshes for elliptic equations. We first derive the global H^1- and L^2-norms error estimates for a general case that the exact solution might be singular, namely, $u \in H^{\frac{3}{2}+\epsilon}$ with $\epsilon > 0$ arbitrarily small. These estimates generalize the existing results that were established under the regularity assumption $u \in H^2$. Then, we establish negative-norm error estimates for solutions with different regularity conditions. Finally, we study the interior estimates to show that the interior error of the FVEMs is bounded by the combination of the best local approximation error and a proper negative-norm error. We provide numerical results to verify our interior estimates.

Key words. finite volume element methods, negative-norm estimates, interior estimates

AMS subject classifications. Primary, 65N30; Secondary, 45N08

DOI. 10.1137/18M1197746

1. Introduction. The finite volume method (FVM) is an important numerical tool solving partial differential equations (PDEs) and enjoys great popularity among engineering computations, especially in computational fluid dynamics. See, for example, [10, 13, 19, 20, 27, 29, 30, 32, 39, 37]. During the past several decades, the study on FVMs has also been an active research area in the computational mathematics community. See [1, 2, 4, 7, 8, 9, 10, 12, 21, 23, 26, 29, 36, 38, 42] and the references cited therein. However, it is a challenging task to set up a systematic theory for the FVM as satisfactory as that for the finite element method (FEM), especially for high-order schemes.

The finite volume element method (FVEM), also known as the box method [1, 16, 36], the generalized finite difference method [23], and the vertex-centered FVM [6], is one of the FVMs which seek the approximate solution in a certain finite element space. With the help of the FE space, the discretization error of the FVEM can be analyzed under the framework of a Petrov–Galerkin scheme [23, 42]. The linear FVEM is closely related to the linear FEM, for which the error analysis has been well established [1, 4, 11, 16, 23, 42]. However, the high-order FVEMs are significantly different from

*Received by the editors July 2, 2018; accepted for publication (in revised form) July 10, 2019; published electronically September 12, 2019.
https://doi.org/10.1137/18M1197746
Funding: The work of the first author was partially supported by the National Natural Science Foundation of China grant 11601536. The work of the second author was partially supported by the National Science Foundation grant DMS-1819041 and by the National Natural Science Foundation of China grant 11628104. The work of the third author was partially supported by the special high performance computing of the National Key Research and Development Program grant 2016YFB0200604, by the National Natural Science Foundation of China grant 11571384, and by the Guangdong Natural Science Foundation grant 2017B030311001.
†School of Data and Computer Science, Sun Yat-sen University, Guangzhou, Guangdong 510006, People's Republic of China (guoli7@mail.sysu.edu.cn).
‡Department of Mathematics, Wayne State University, Detroit, MI 48202 (li@wayne.edu).
§Corresponding author. School of Data and Computer Science, and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou, Guangdong 510006, People's Republic of China (mcszqs@mail.sysu.edu.cn).

high-order FEMs and it is more challenging to obtain the error estimate. The early efforts on high-order FVEMs can be traced back to [40, 24, 23, 5, 31, 42, 41]. Recently, a class of high-order FVE schemes over quadrilateral meshes has been designed and analyzed in [43]. In this class of FVEMs, the control volume was constructed using Gauss points in each quadrilateral element and optimal error analysis in the global energy norm was derived. Then in [25] and [17], the L^2- and L^∞-norm errors are also shown to have optimal convergence under appropriate assumptions on the solution regularity and on the underlying meshes.

As a continuation of [43, 22, 15, 25, 17], in this paper we study interior H^1- and L^2-norm error estimates for high-order FVEMs. To illustrate our ideas and techniques, we will consider the FVEMs for the following model problem:

$$-\nabla \cdot (\nu \nabla u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, f a given real valued function, and the coefficient matrix $\nu \in (L^\infty(\Omega))^{2\times 2}$ is uniformly bounded and positive definite in the sense that that there exists two constant $\nu_0, \nu_1 > 0$ such that for all $\xi \in \mathbb{R}^2$, there holds $\nu_0\|\xi\|^2 \le \xi^T \nu \xi \le \nu_1 \|\xi\|^2$.

Note that interior estimates of the FEMs for (1.1) have been investigated in a series of papers (see, e.g., [28, 33, 35, 34]). For the FEMs, the interior error in an interior region of the domain is bounded by the local approximation error and the FE negative-norm error in the global domain. It is also known that the FE negative-norm estimate has been thoroughly studied in the literature (see, e.g., [3]).

The main difficulty in the analysis of the FVEM interior error is that the FVEM bilinear form is not symmetric and does not induce an inner product naturally. Therefore, the duality argument used in the negative-norm and interior estimates of the FEM does not apply to the FVEM. To overcome this difficulty, we shall estimate the difference between the FVEM and FEM bilinear forms and the difference between the corresponding right-hand sides. Starting with the ideas in [25, 15], we will develop new analytical techniques to obtain negative-norm estimates and interior estimates for the FVEM.

Note that the solution of (1.1) may be singular near the nonsmooth points on the boundary, while the FVEM theory in the literature was established under the assumption that solution u is sufficient regular. For instance, $u \in H^2$ is the minimum requirement for most of the exiting results. Thus, we first generalize the current *global* FVEM error estimate to the case when u is singular. With the help of the trace theorem [14], we obtain convergence results for the case $u \in H^{3/2+\epsilon}$ with $\epsilon > 0$ arbitrary small. Consequently, our negative-norm estimates for the FVEM will be established under low-regularity assumptions.

For the interior error estimate of the FVEM, we follow two main steps. In the first step, we estimate a discrete version of (1.1) given by the FVEM in an interior domain G_0. One important result is that the H^1- and L^2-norms of this discrete error are bounded by its negative-norms. In the second step, we bound the H^1- and L^2-norms of the FVEM error in an interior domain G_0 by the approximate error in a slightly larger domain G and the negative-norm of the error.

The findings in this paper are important for the theoretical development of the FVEMs. First, all the current estimates for the FVEMs are based on the assumption that the exact solution u at least belongs to H^2. Here, we allow less-regular solutions $u \in H^{\frac{3}{2}+\epsilon}, \epsilon > 0$ arbitrary small, to derive optimal H^1- and L^2-error estimates. This makes the estimates of the FVEM available for singular problems. Second, to the best of our knowledge, it is the first time that the negative-norm and interior error estimates are obtained for high-order FVEMs.

The organization of this paper is as follows. In section 2, we generalize the global H^1- and L^2-estimates to equations with low-regularity solutions. In section 3, we present two negative-norm estimates for the FVEM, one under the high-regularity assumption and the other under low-regularity assumption on the equation. The interior estimates are included in section 4. We provide numerical test results in section 5 to verify the theory. Concluding remarks are given in section 6.

Throughout the paper, we denote the standard Sobolev space on Ω by $H^m(\Omega)$ if m is an integer, and let $L^2 = H^0$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. For a noninteger $\gamma \geq 0$, let the space H^γ and H_0^γ be defined by interpolation. We use the notation $\|\cdot\|_\alpha = \|\cdot\|_{\alpha,\Omega}$ for $\alpha \in \mathbb{R}$ if Ω is the underlying domain. For two regions A and B, $A \subset B$ means that A is an interior proper subset of B (i.e., $\text{dist}(\partial A, \partial B) > 0$), while $A \subseteq B$ means that A is a subset of B (A can be equal to B). In addition, "$\alpha \lesssim \beta$" means that α can be bounded by β multiplied by a constant which depends on the underlying domain and the coefficient matrix ν, but not on the functions or the mesh size involved in the estimates. Meanwhile, "$\alpha \sim \beta$" means "$\alpha \lesssim \beta$" and "$\beta \lesssim \alpha$".

2. Global error estimates in H^1- and L^2-spaces.

In this section, we generalize the global H^1- and L^2-norm error estimates of high-order FVEMs in [25, 43] (under high-regularity assumptions on the exact solution) to the case that the solution has lower regularity.

2.1. High-order FVEMs.

We recall a class of FVEM introduced and investigated in [43]. Let $\mathcal{T}_h = \{\tau\}$ be a quasi-uniform quadrilateral partition of Ω, where $h = \max_{\tau\in\mathcal{T}_h}(\text{diam}\tau)$ is the mesh parameter. Denote the set of all vertices and all edges of $\mathcal{T}_h$ by $\mathcal{N}_h$ and $\mathcal{E}_h$, respectively. Moreover, let $\mathcal{N}_h^\circ = \mathcal{N}_h \setminus \partial\Omega$, $\mathcal{E}_h^\circ = \mathcal{E}_h \setminus \partial\Omega$, $\mathcal{N}_h^b = \mathcal{N}_h \cap \partial\Omega$, and $\mathcal{E}_h^b = \mathcal{E}_h \cap \partial\Omega$ be the set of interior vertices, internal edges, boundary vertices, and boundary edges, respectively. Throughout the whole paper, we suppose that the quadrilateral mesh $\mathcal{T}_h$ is always of $\mathcal{O}(h^2)$-distortion from a parallel mesh in the sense that the distance between midpoints of two diagonals of each $\tau \in \mathcal{T}_h$ is bounded by $\mathcal{O}(h^2)$ (cf. [25, 41]), and we suppose that the coefficient matrix ν is in $(W^{2k}(\Omega))^{2\times 2}$ piecewisely with respect to $\mathcal{T}_h$.

Define the continuous finite element space of all bi-k polynomials associated with $\mathcal{T}_h$

$$S_h = S_h^k(\Omega) = \{v \in C(\Omega) \big| \hat{v}_\tau = v \circ F_\tau \in \mathbb{Q}^k(\hat{\tau})\ \forall \tau \in \mathcal{T}_h, \text{and } v|_{\partial\Omega} = 0\},$$

where $\mathbb{Q}^k(\hat{\tau})$ is the space of all bi-k polynomials and F_τ is the bilinear transformation from the reference $\hat{\tau} = [-1,1]^2$ to τ. Then, the FEM for (1.1) is to find $u_{h,FE} \in S_h$, such that

$$a(u_{h,FE}, \chi) = (\nu \nabla u_{h,FE}, \nabla \chi) = (f, \chi) \quad \forall \chi \in S_h, \tag{2.1}$$

where $(v, w) = \int_\Omega vwdxdy$ is the L^2 inner product and $a(\cdot,\cdot)$ is the FEM bilinear form.

A class of FVE schemes can be defined as follows. For each $\tau \in \mathcal{T}_h$, let F_τ be the affine transformation from the reference $\hat{\tau} = [-1,1]^2$ to τ. Let $\{g_i | i = 1,\ldots,k\}$ be the collection of the k Gauss points, i.e., zeros of L_k (the Legendre polynomial of degree k), on the interval $[-1,1]$; let $\{l_j | j = 0,\ldots,k\}$ be the collection of the $k+1$ Lobatto points of degree k in the interval $[-1,1]$. Namely, $l_0 = -1, l_k = 1$, and $\{l_m | m = 1,\ldots,k-1\}$ are the $k-1$ zeros of L_k'. We define the sets of the Gauss points and the Lobatto points in τ by $\mathcal{G}_\tau = \{g^\tau_{i,j} = F_\tau(g_i, g_j) | i, j \in \{1, \dots, k\}\}$ and $\mathcal{L}_\tau = \{L^\tau_{i,j} = F_\tau(l_i, l_j) | i, j \in \{0, 1, \dots, k\}\}$, respectively. Moreover, let $\mathcal{G} = \cup_{\tau\in\mathcal{T}_h}\mathcal{G}_\tau$ and $\mathcal{L} = \cup_{\tau\in\mathcal{T}_h}\mathcal{L}_\tau$ be the sets of all Gauss and Lobatto points in $\mathcal{T}_h$, respectively. The FVE scheme is designed by constructing the so-called *dual mesh* associated with the Gauss points. We decompose each $\tau \in \mathcal{T}_h$ into $(k+1)^2$ subquadrilaterals $\tau_P, P \in \mathcal{G}_\tau$, by connecting each Gauss point on one edge of τ to the one at the same position on its opposite edge. For any given Lobatto point $P \in \mathcal{L}$, a control volume V_P is constructed as the union of all subquadrilaterals containing the node P. The collection of all control volumes $\mathcal{T}_h^* = \{V_P | P \in \mathcal{L}\}$ constitutes the dual mesh of Ω.

The FVE solution of (1.1) is a function $u_h \in S_h$ satisfying

$$-\int_{\partial V_P} \nu \nabla u_h \cdot \mathbf{n} ds = \int_{V_P} f dx dy \tag{2.2}$$

on each control volume $V_P, P \in \mathcal{L}^\circ$, where $\mathbf{n}$ is the unit outward normal on the boundary curve ∂V_P. Define the test space

$$\mathcal{V}_h = \mathcal{V}_h(\Omega) = \mathrm{span}\{\psi_{V_P} | P \in \mathcal{L}^\circ\},$$

where $\mathcal{L}^\circ = \mathcal{L}\setminus\partial\Omega$ is the set of all interior Lobatto points and ψ_A is the characteristic function of some set $A \subset \Omega$ defined by $\psi_A(x) = 1$ if $x \in A$ and $\psi_A(x) = 0$ if $x \in \Omega \setminus A$. Then, (2.2) can be rewritten in the following Petrov–Galerkin form:

$$a_h(u_h, w_h) = (f, w_h) \quad \forall w_h \in \mathcal{V}_h, \tag{2.3}$$

where the FVEM bilinear form is defined for all $v \in H_0^1(\Omega), w_h = \sum_{P\in\mathcal{L}^\circ} w_P \psi_{V_P} \in \mathcal{V}_h$ as

$$a_h(v, w_h) = -\sum_{p\in\mathcal{L}^\circ} w_P \int_{\partial V_P} \nu \nabla v \cdot \mathbf{n} ds. \tag{2.4}$$

Denoting by $\mathcal{E}_h'$ the set of interior edges of the dual partition $\mathcal{T}_h^*$, the bilinear form $a_h(\cdot,\cdot)$ can be rewritten as

$$a_h(v, w_h) = \sum_{E\in\mathcal{E}_h'} [w_h]_E \int_E \nu \frac{\partial v}{\partial \mathbf{n}} ds \quad \forall v \in H_0^1(\Omega),\ w_h \in \mathcal{V}_h, \tag{2.5}$$

where $[w_h]_E = w_h\big|_{V_2} - w_h\big|_{V_1}$ denotes the jump of the w_h across the common edge $E = V_1 \cap V_2$ of two volumes $V_1, V_2 \in \mathcal{T}_h^*$ and $\mathbf{n}$ denotes the normal vector on E pointing from V_1 to V_2.

2.2. H^1- and L^2-norm error estimates. With the form (2.3), the convergence properties of the FVEM can be established under the framework of a Petrov–Galerkin method. Namely, the FVEM error can be estimated by studying the continuity and the inf-sup condition of the FVEM bilinear form $a_h(\cdot,\cdot)$ (cf. [42]).

Along this direction, it is shown in [43] that when $\mathcal{T}_h$ is sufficiently shape regular, there holds the following inf-sup condition:

$$\inf_{v_h\in S_h} \sup_{w_h\in\mathcal{V}_h} \frac{a_h(v_h, w_h)}{|v_h|_1 |w_h|_h'} \gtrsim 1, \tag{2.6}$$

where the seminorm in the test space $\mathcal{V}_h$ is defined by

$$|w_h|_h^{'} = \left(\sum_{E\in\mathcal{E}_h'} h_E^{-1} \int_E [w_h]_E^2 ds \right)^{\frac{1}{2}}$$

with h_E as the diameter of an edge E. We mention that by defining a *from-trial-to-test* mapping Π (cf. [43, 25] for a detailed definition), the inf-sup condition (2.6) is equivalent to the coercivity

$$|a_h(v_h, v_h^*)| \gtrsim |v_h|_1^2 \quad \forall v_h \in S_h, \tag{2.7}$$

where we have used the notation $v_h^* = \Pi v_h \in \mathcal{V}_h$ and the equivalence $|v_h|_1 \sim |v_h^*|_h^{'}$.

For the continuity, it is shown in [42] that

$$|a_h(v, w_h)| \lesssim |v|_h |w_h|_h^{'} \tag{2.8}$$

holds for all $v \in H_0^1(\Omega) \cap H_h^2(\Omega)$ and $w_h \in \mathcal{V}_h$, where the seminorm $|\cdot|_h$ in the broken space $H_h^2(\Omega) = \{v \in H^1(\Omega) : v|_\tau \in H^2 \text{ for all } \tau \in \mathcal{T}_h\}$ is defined as $|v|_h = \left(\sum_{\tau\in\mathcal{T}_h} |v|_{1,\tau}^2 + h_\tau^2 |v|_{2,\tau}^2\right)^{\frac{1}{2}}$ with h_τ as the diameter of τ. Here we would like to point out that in fact, the inequality (2.8) holds for functions in a larger broken Sobolev space

$$H_h^\alpha = H_h^\alpha(\Omega) = \{v \in H^1(\Omega) : v|_\tau \in H^\alpha \ \forall \tau \in \mathcal{T}_h\}, \quad \alpha > \frac{3}{2},$$

with the associated seminorm $|v|_{h,\alpha} = (\sum_{\tau\in\mathcal{T}_h}(|v|_{1,\tau}^2 + h_\tau^{2(\alpha-1)}|v|_{\alpha,\tau}^2))^{\frac{1}{2}}$ for all $v \in H_h^\alpha$. Namely, we have the following lemma.

LEMMA 2.1. *The inequality* (2.8) *holds for all* $v \in H_0^1(\Omega) \cap H_h^\alpha$ $(\alpha > \frac{3}{2})$ *and* $w_h \in \mathcal{V}_h$ *with* $|\cdot|_h$ *replaced by* $|\cdot|_{\alpha,h}$.

Proof. By the Cauchy–Schwartz inequality, we have

$$|a_h(v, w_h)| = \left| \sum_{E\in\mathcal{E}_h'} [w_h]_E \int_E \nu \frac{\partial v}{\partial \mathbf{n}} ds \right| \lesssim |w_h|_h' \left(\sum_{E\in\mathcal{E}_h'} h_E \int_E \left(\nu \frac{\partial v}{\partial \mathbf{n}}\right)^2 ds \right)^{\frac{1}{2}}.$$

By Theorem 1.4.2 in [14], for all $\tau \in \mathcal{T}_h, v \in H^\beta(\tau), \beta > \frac{1}{2}$, there holds

$$|v|_{0,\partial\tau} \lesssim h_\tau^{-\frac{1}{2}} |v|_{0,\tau} + h_\tau^{\beta-\frac{1}{2}} |v|_{\beta,\tau},$$

where the hidden constant depends only on the shape of τ. Choosing $\beta = \alpha - 1$ and combining with the fact that each entry of ν is bounded by a constant, we have

$$|a_h(v, w_h)| \lesssim |w_h|_h' \left(\sum_{E\in\mathcal{E}_h'} \sum_{E\cap\tau\neq\emptyset} (|\nabla v|_{0,\tau}^2 + h_\tau^{2\beta} |\nabla v|_{\beta,\tau}^2) \right)^{\frac{1}{2}} \lesssim |v|_{h,\alpha} |w_h|_h^{'}, \tag{2.9}$$

which completes the proof. ∎

With the inf-sup condition (2.6) and the generalized continuity (2.9), it is easy to show the following H^1 error estimates.

THEOREM 2.2. *Let $u \in H^{1+s}(\Omega)$ ($\frac{1}{2} < s \le k$) and $u_h \in S_h$ be the exact solution and the FVEM solution of* (1.1)*, respectively. Then*

$$\|u - u_h\|_1 \lesssim h^s \|u\|_{s+1}. \tag{2.10}$$

Proof. Let $u_I \in S_h$ be the nodal interpolation of u. According to (2.9) and (2.7), we have

$$\|u_I - u_h\|_1^2 \lesssim a_h(u_I - u_h, u_I^* - u_h^*) = a_h(u_I - u, u_I^* - u_h^*) \lesssim |u - u_I|_{h,1+s}\|u_I - u_h\|_1.$$

Then, we obtain by the triangle inequality,

$$\begin{aligned} \|u - u_h\|_1 &\le \|u - u_I\|_1 + \|u_I - u_h\|_1 \lesssim \|u - u_I\|_1 + |u - u_I|_{h,s+1} \\ &\lesssim \|u - u_I\|_1 + \left(\sum_{\tau \in \mathcal{T}_h} h^{2s}|u - u_I|_{s+1,\tau}^2\right)^{1/2} \lesssim h^s\|u\|_{s+1}, \end{aligned} \tag{2.11}$$

which completes the proof. ∎

Remark 2.3. Theorem 2.2 generalizes the estimate in [25, 43], where (2.10) was shown only for positive integer s. Here (2.10) holds for any real number $1/2 < s \le k$.

For the L^2 error estimate, an *optimal order* estimate has been given in [25] under the assumption that $u \in H^{k+1}(\Omega)$ and $f \in H^k(\Omega)$. In this paper, we extend this result to the more general case in which $u \in H^{1+s}(\Omega)$ and $f \in H^s(\Omega)$ $(1/2 < s \le k)$. To this end, we introduce an index t that is determined by the geometry of the domain Ω. For any sufficiently smooth function $g \in C^\infty(\Omega)$, let ψ be the solution of the problem

$$-\nabla \cdot (\nu \nabla \psi) = g \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega. \tag{2.12}$$

The index $t > 0$ is such that

$$\|\psi\|_{t+1} \lesssim \|g\|_{t-1} \tag{2.13}$$

holds for all $g \in C^\infty(\Omega)$. In fact, let θ be the largest interior angle of Ω. Then, for all $0 < t < \pi/\theta$, the inequality (2.13) holds. We point out that for (1.1), the regularity index s of the solution u depends on both the right-hand function f and the geometry of Ω. When the boundary of polygonal domain has large interior corners, even when f is smooth, the full regularity property $\|u\|_{k+1} \lesssim \|f\|_{k-1}$ may not be valid. We also mention that the regularity of u is a local property. In an interior region, the index s only depends on f, in other words, when f is sufficiently smooth, the solution u can display a full regularity in an interior region of Ω; see section 4 for details.

Following the routines in [25], we obtain the global L^2 error estimate.

THEOREM 2.4. *Let $u \in H^{1+s}(\Omega)$ with $\frac{1}{2} < s \le k$ and $u_h \in S_h$ be the exact solution and the FVEM solution* (1.1)*, respectively. If $f \in H^s(\Omega)$, then*

$$\|u - u_h\|_0 \lesssim h^{s+\min(1,t)}(\|u\|_{s+1} + \|f\|_s). \tag{2.14}$$

Remark 2.5. Comparing to the L^2 estimates for the FEM, we observe that the L^2 error estimate for the FVEM has an additional regularity requirement for f. For the special case $k = s = 1$, a counterexample has been designed in [18] to illustrate that $u \in H^{1+s}$ is not sufficient to obtain the optimal order L^2 estimates for the FVEM and the additional term $\|f\|_s$ on the right-hand side of (2.14) is necessary.

Remark 2.6. The L^2 estimate in [25] is a special case of Theorem 2.4 where $s = k$ and $t = 1$.

3. Negative-norm error estimates. We begin with a definition of negative norms. For any subset $A \subseteq \Omega$ and any integer $m \geq 0$, the negative norm is defined as

$$\|v\|_{-m,A} = \sup_{0 \neq \varphi \in C_0^\infty(A)} \frac{(v,\varphi)_A}{\|\varphi\|_{m,A}},$$

where $(v,\varphi)_A = \int_A v\varphi dxdy$.

LEMMA 3.1. *For all* $v \in H_h^{k+l+1}, 0 \leq l \leq k$ *and* $\xi_h \in S_h^k(\Omega)$, *we have*

$$|a(v,\xi_h) - a_h(v,\xi_h^*)| \lesssim h^{l+k}\|v\|_{k+l+1,h}\|\xi_h\|_{k+1,h}. \tag{3.1}$$

Proof. Following Theorem 4.4 in [25], we rewrite the FVE bilinear form $a_h(\cdot,\Pi\cdot)$ as a Gauss quadrature of the FE bilinear form $a(\cdot,\cdot)$ and then

$$a(\xi_h,v) - a_h(\xi_h,v^*) = \sum_{Q\in\mathcal{T}_h} (a_Q(\xi_h,v) - a_{h,Q}(\xi_h,v^*)) \tag{3.2}$$

with

$$\begin{aligned} a_{h,Q}(\xi_h,v^*) - a_Q(\xi_h,v) = \int_{-1}^1 E_1(\Theta_1,\hat{y})d\hat{y} + \sum_{i=1}^k A_i E_2(\Theta_1,g_i) \\ + \int_{-1}^1 E_2(\Theta_2,\hat{x})d\hat{x} + \sum_{j=1}^k A_j E_1(\Theta_2,g_j); \end{aligned}$$

here A_i, g_i are the weights and abscissae of the k-point-Gauss-quadrature for computing the integral $\int_{-1}^1 z(x)dx$,

$$E_1(F,\hat{y}) = \int_{-1}^1 F(\hat{x},\hat{y})d\hat{x} - \sum_{i=1}^k A_i F(g_i,\hat{y}), E_2(F,\hat{y}) = \int_{-1}^1 F(\hat{x},\hat{y})d\hat{y} - \sum_{j=1}^k A_j F(\hat{x},g_j),$$

and $\Theta_i = \hat{\Theta}_{i,Q} = \frac{\partial^2\hat{\xi}_h}{\partial\hat{x}\partial\hat{y}}\hat{V}_i$, $i = 1,2$ with

$$\begin{aligned} \hat{V}_1(\hat{x},\hat{y}) &= \hat{V}_1(-1,\hat{y}) + \int_{-1}^{\hat{x}} \hat{\nu}\left(b_{12}\frac{\partial\hat{v}}{\partial\hat{x}} + b_{11}\frac{\partial\hat{v}}{\partial\hat{y}}\right)(\hat{x}',\hat{y})d\hat{x}', \\ \hat{V}_2(\hat{x},\hat{y}) &= \hat{V}_2(\hat{x},-1) + \int_{-1}^{\hat{y}} \hat{\nu}\left(b_{22}\frac{\partial\hat{v}}{\partial\hat{x}} + b_{21}\frac{\partial\hat{v}}{\partial\hat{y}}\right)(\hat{x},\hat{y}')d\hat{y}'. \end{aligned}$$

The matrix $b = ((-1)^{i+j}b_{i,j})_{2\times2}$ and $(b_{i,j})_{2\times2} = J_Q^{-1}(DF_Q)(DF_Q)^T$ satisfies

$$|D^{\mathbf{i}}b_{lm}| \lesssim h^{|\mathbf{i}|}, \quad 1 \leq l,m \leq 2, \tag{3.3}$$

where $\mathbf{i} = i_1, i_2,\ i_1, i_2 \geq 0$ and $|\mathbf{i}| = i_1 + i_2$ and DF_Q is the Jacobi matrix of F_Q and J_Q is the determinant of DF_Q.

Next we estimate $E_1(\Theta_1,\hat{y})$, $E_2(\Theta_2,\hat{x})$, $E_2(\Theta_1,g_i)$, and $E_1(\Theta_2,g_j)$. For any given $\hat{y} \in [-1,1]$, $E_1(\Theta_1,\hat{y})$ is the difference between an exact integral and a Gauss quadrature of order k, and then we have

$$|E_1(\Theta_1,\hat{y})| \lesssim \left\|\frac{\partial^{k+l}}{\partial\hat{x}^{k+l}}\Theta_1(\cdot,\hat{y})\right\|_{\hat{Q}}.$$

By the Leibnitz formula, it is easy to obtain

$$\frac{\partial^{k+l}}{\partial \hat{x}^{k+l}}\Theta_1 = \frac{\partial^{k+l}}{\partial \hat{x}^{k+l}}\left(\frac{\partial^2 \hat{\xi}_h}{\partial \hat{x}\partial \hat{y}}\hat{V}_1\right) = \sum_{j=0}^{k-1} C_{k+l}^j \frac{\partial}{\partial \hat{y}}\left(\frac{\partial^{j+1}\hat{\xi}_h}{\partial \hat{x}^{j+1}}\right)\frac{\partial^{k+l-j}}{\partial \hat{x}^{k+l-j}}\hat{V}_1. \tag{3.4}$$

Using the Leibnitz formula again, we have

$$\begin{aligned} \left|\frac{\partial^{k+l-j}}{\partial \hat{x}^{k+l-j}}\hat{V}_1\right| &= \left|\frac{\partial^{k+l-j-1}}{\partial \hat{x}^{k+l-j-1}}\left(\hat{\nu}\frac{\partial \hat{v}}{\partial \hat{x}}b_{12} + \hat{\nu}\frac{\partial \hat{v}}{\partial \hat{y}}b_{11}\right)\right| \\ &\lesssim \sum_{m=0}^{k+l-j-1}\left|\left(\frac{\partial^m(\hat{\nu}\frac{\partial \hat{v}}{\partial x})}{\partial \hat{x}^m}\frac{\partial^{k+l-j-1-m}b_{12}}{\partial \hat{x}^{k+l-j-1-m}} + \frac{\partial^m(\hat{\nu}\frac{\partial \hat{v}}{\partial y})}{\partial \hat{x}^m}\frac{\partial^{k+l-j-1-m}b_{11}}{\partial \hat{x}^{k+l-j-1-m}}\right)\right| \\ &\lesssim \sum_{m=0}^{k+l-j-1} h^{k+l-j-1-m}\left(\left|\hat{\nu}\frac{\partial \hat{v}}{\partial \hat{x}}\right|_{m,\hat{Q}} + \left|\hat{\nu}\frac{\partial \hat{v}}{\partial \hat{y}}\right|_{m,\hat{Q}}\right) \lesssim h^{k+l-j-1}\|v\|_{k+l,Q}, \end{aligned} \tag{3.5}$$

where in the last inequality, we have used the chain rule, $\|\nu\|_{2k,\infty} \lesssim 1$, and the fact that

$$|\hat{\chi}|_{l,\hat{Q}} \lesssim h^{l-1}\|\chi\|_{l,Q} \ \forall \chi \in H^l(Q), l \geq 1.$$

Consequently,

$$\begin{aligned} |E_1(\Theta_1,\hat{y})| &\lesssim \sum_{j=0}^{k-1} h^{j+1}\|\xi_h\|_{j+2,Q} \sum_{m=0}^{k+l-j-1} h^{k+l-j-1}\|v\|_{m+1,Q} \\ &\lesssim \sum_{j=0}^{k-1}\sum_{m=0}^{k+l-j-1} h^{k+l}\|v\|_{m+1,Q}\|\xi_h\|_{j+2,Q} \\ &\lesssim h^{k+l}\|\xi_h\|_{k+1,Q}\|v\|_{k+l,Q}. \end{aligned}$$

Since here $\hat{x} \in (0,1)$ is arbitrary, actually we obtain

$$\|E_1(\Theta_1,\cdot)\|_\infty \lesssim h^{k+l}\|\xi_h\|_{k+1,Q}\|v\|_{k+l,Q}. \tag{3.6}$$

Similarly, we have

$$\|E_1(\Theta_2,\cdot)\|_\infty \lesssim h^{k+l}\|\xi_h\|_{k+1,Q}\|v\|_{k+l+1,Q}. \tag{3.7}$$

For E_2, we also have $\|E_2(\Theta_1,\cdot)\|_\infty, \|E_2(\Theta_2,\cdot)\|_\infty \lesssim h^{k+l}\|\xi_h\|_{k+1,Q}\|v\|_{k+l+1,Q}$. Plugging the estimates (3.6) and (3.7) into (3.2), we obtain (3.1). The proof is completed. ∎

With this estimate for the difference between the FVEM and FEM bilinear forms, we first derive a negative-norm estimate under a high regularity assumption on the solution.

THEOREM 3.2. *Let $u \in H^{k+l+1}$ $(0 \leq l \leq k)$ and u_h be the exact solution and the FVEM solution, respectively. Then for any integer $0 \leq p \leq k-1$, we have*

$$\|u - u_h\|_{-p} \lesssim h^{l+\min(t,p+1)}\|u\|_{k+l+1}. \tag{3.8}$$

Proof. For simplicity, we denote $e = u - u_h$. By (2.13), we obtain

$$\|e\|_{-p} = \sup_{g\in C_0^\infty(\Omega)}\frac{(e,g)}{\|g\|_p} \lesssim \sup_{\psi\in C^\infty}\frac{a(e,\psi)}{\|\psi\|_{1+\min(t,p+1)}}. \tag{3.9}$$

Letting $\psi_I \in S_h^k(\Omega)$ be the nodal interpolation of ψ, one obtains

$$a(e, \psi) = a(e, \psi - \psi_I) + a(e, \psi_I). \tag{3.10}$$

For the first term of the right-hand side (3.10), we have

$$a(e, \psi - \psi_I) \lesssim \|e\|_1 \|\psi - \psi_I\|_1 \lesssim h^{k+\min(t,p+1)} \|u\|_{k+1} \|\psi\|_{1+\min(t,p+1)}. \tag{3.11}$$

For the second right term of (3.10), we use the fact that

$$a_h(e, \psi_I^*) = a_h(u, \psi_I^*) - a_h(u_h, \psi_I^*) = 0,$$

the Theorem 2.2, (3.1), and the inverse inequality (see [28])

$$\|v_h\|_q \lesssim h^{p-q} \|v_h\|_p \; \forall v_h \in S_h^k(\Omega), p \le q \le k,$$

to derive that

$$\begin{aligned} |a(e, \psi_I)| = |a(e, \psi_I) - a_h(e, \psi_I^*)| &\lesssim h^{l+k} \|u - u_h\|_{k+l+1,h} \|\psi_I\|_{k+1,h} \\ &\lesssim h^{l+\min(t,p+1)} \|u\|_{k+l+1} \|\psi\|_{1+\min(t,p+1)}. \end{aligned} \tag{3.12}$$

Plugging (3.11) and (3.12) into (3.10) and (3.9), we obtain (3.8). □

Note that for the FEM, the optimal negative-norm estimate is $\|e\|_{-(k-1)} \lesssim h^{k+\min(k,t)} \|u\|_{k+1}$, ([3]). Here for the FVEM, we require the regularity $u \in H^{2k+1}$ for the optimal order $k + \min(k, t)$. Since the elliptic problems considered here may have singular solutions, we next present a result in which u has singularities.

THEOREM 3.3. *Let $u \in H^{s+1}$ with $\frac{1}{2} < s \le 1$ and u_h be the exact solution of* (1.1) *and the FVEM approximation solution, respectively. Then for any integer $0 \le p \le k - 1$, if $f \in H^{\min(t,p+1)}$, we have*

$$\|u - u_h\|_{-p} \lesssim h^{s+\min(t,p+1)} (\|u\|_{1+s} + \|f\|_{\min(t,p+1)}). \tag{3.13}$$

Proof. Following the estimates in Theorem 3.2, we only need to bound the two terms on the right-hand side of (3.10). For the first term, because of (2.10), we have

$$|a(e, \psi - \psi_I)| \lesssim |e|_1 |\psi - \psi_I|_1 \lesssim h^{s+\min(t,p+1)} \|u\|_{1+s} \|\psi\|_{1+\min(t,p+1)}. \tag{3.14}$$

For the second term, we have that

$$\begin{aligned} a(e, \psi_I) &= a(u - u_h, \psi_I) - a_h(u - u_h, \psi_I^*) \\ &= a_h(u_h, \psi_I^*) - a(u_h, \psi_I) + (f, \psi_I - \psi_I^*). \end{aligned} \tag{3.15}$$

By (3.1) and the inverse inequality, one has

$$|a_h(u_h, \psi_I^*) - a(u_h, \psi_I)| \lesssim h^{s+\min(t,p+1)} \|u\|_{1+s} \|\psi\|_{1+\min(t,p+1)}. \tag{3.16}$$

Letting $f_I \in S_h^{k-1}(\Omega)$ be the interpolation of f, one has (cf. Theorem 4.7 in [25])

$$\begin{aligned} &|(f, \psi_I - \psi_I^*)| \\ &\quad = |(f - f_I, \psi_I - \psi_I^*)| \lesssim \|f - f_I\|_0 \|\psi_I - \psi_I^*\|_0 \\ &\quad \lesssim h |\psi_I|_1 h^{\min(t,p+1)} \|f\|_{\min(t,p+1)} \lesssim h^{1+\min(t,p+1)} \|f\|_{\min(t,p+1)} \|\psi\|_{1+\min(t,p+1)}. \end{aligned} \tag{3.17}$$

Combining (3.14)–(3.17), we derive the desired inequality (3.13). □

Remark 3.4. In Theorem 3.3, we have an additional requirement $f \in H^{\min(t,p+1)}$. Since the regularity of u depends on both f and the geometry of Ω, u might have singularity even if $f \in C^\infty$. The requirement $f \in H^{\min(t,p+1)}$ does not contradict the low regularity assumption $u \in H^{1+s}$.

4. Interior error estimates. In this section, we investigate the interior error estimates in the H^1 and L^2 spaces for the FVE scheme (2.3). We begin with some necessary notation. For an interior domain $A \subset \Omega$, let $S_h(A)$ and $\mathcal{V}_h(A)$ be the restriction on A of $S_h(\Omega)$ and $\mathcal{V}_h(\Omega)$, respectively. Moreover, let $\mathring{S}_h(A) = \{\chi \in S_h(A) : \text{supp } \chi \subseteq A\}$, and $\mathring{\mathcal{V}}_h(A) = \{\chi \in \mathcal{V}_h(A) : \text{supp } \chi \subseteq A\}$. It is clear that $\mathring{S}_h(A) \subset S_h$ and $\mathring{\mathcal{V}}_h(A) \subset \mathcal{V}_h$ for all $A \subset \Omega$.

Suppose now $D_0 = B(x_0, r_0) \subset D = B(x_0, r) \subset \Omega$ are two fixed concentric discs with radii r_0 and r, respectively. We also assume that the mesh size h is sufficiently small, such that for any concentric discs between D_0 and D, the distance between their boundaries for any two adjacent discs is larger than $k_0 h$, where k_0 is a positive constant. Let $\omega \in C_0^\infty(D_0)$, and then for all $\chi \in S_h(D)$, there exists $\zeta \in \mathring{S}_h(D)$, such that the following superapproximation holds for $0 \le \eta \le 1$

$$\|\omega\chi - \zeta\|_{\eta,D} \lesssim h^{2-\eta}\|\chi\|_{1,D}. \tag{4.1}$$

The inequality (4.1) can be derived by using Theorem 4.6.11 in [3] for the case that η is an integer. And there is an example to verify this inequality in [28]. Note that the hidden constant in (4.1) may depend on the center x_0 and the radii r_0 and r.

By the proof of Lemma 3.1, we find that the inequality (3.1) is also valid on an interior region of the domain Ω. In particular, letting $D_0 \subset D \subset \Omega$ be two arbitrary but fixed concentric discs with $\text{dist}(\partial D_0, \partial D) \gtrsim h$, we have

(4.2)
$$|a(\xi_h, v_h)_{D_0} - a_h(\xi_h, v_h^*)_{D_0}| \lesssim h^{2k}\|\xi_h\|_{k+1,h,D}\|v_h\|_{k+1,h,D} \lesssim h^k\|\xi_h\|_{1,h,D}\|v_h\|_{k+1,h,D},$$

where $a(\cdot,\cdot)_{D_0}$, $a_h(\cdot,\cdot)_{D_0}$ and $\|\cdot\|_{k+1,h,D}$ are restrictions of $a(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ on the subset D_0 and $\|\cdot\|_{k+1,h}$ on D.

Next, we introduce the local FVE projection $R_h = R_h^A$ which maps any function $v \in H_0^1(A_0)$ and $v = 0$ in $A \setminus A_0$ (where $A_0 \subset A$ and $\text{dist}(\partial A_0, \partial A) \ge k_0 h$) to $R_h v \in \mathring{S}_h(A)$ such that

$$a_h(R_h v, w_h) = a_h(v, w_h) \quad \forall w_h \in \mathring{\mathcal{V}}_h(A). \tag{4.3}$$

Note that when $A = D$ is a disc in the interior of Ω, the boundary of D is sufficiently smooth, and the full regularity property $\|\psi\|_{k+2,D} \lesssim \|g\|_{k,D}$ holds for all $\psi \in H_0^1(D)$ satisfying the equation

$$-\nabla \cdot (\nu \nabla \psi) = g \quad \text{in} \quad D, \qquad \psi = 0 \quad \text{on} \quad \partial D. \tag{4.4}$$

Therefore, by Theorem 2.2, for all $v \in H_0^1(D_0) \bigcap H^{1+s}(D)$ and $v = 0$ in $D \setminus D_0$ with $\frac{1}{2} < s \le k$, we have

$$\|v - R_h v\|_{1,D} \lesssim h^s \|v\|_{1+s,D}. \tag{4.5}$$

Furthermore, since $\|-\Delta v\|_{s,D} \le \|v\|_{s+2,D}$, by Theorem 2.4, for $v \in H_0^1(D_0) \bigcap H^{2+s}(D)$ and $v = 0$ in $D \setminus D_0$ with $\frac{1}{2} < s \le k$, we have

$$\|v - R_h v\|_{0,D} \lesssim h^{s+1}\|v\|_{s+2,D}. \tag{4.6}$$

4.1. Estimates of the discrete FVE interior error. The aim of this paper is to estimate the FVE error $e = u - u_h$ on an interior domain $G_0 \subset G \subset \Omega$ with $d_0 = \text{dist}(\partial G_0, \partial G) \gtrsim h$. Note that $\bar{G}_0$ can be covered by a finite number of discs

$D_0(x_i, d_0/2)$ with center x_i and radius $d_0/2$, where $\bar{G}_0$ is the closure of G_0. Therefore, it is sufficient to derive interior estimates on each disc $D_0(x_i, d_0/2)$ in order to obtain the interior estimates on the domain G_0. Thus, without loss of generality, we only need to consider the local estimate on an interior disc $D_0 \subset D \subset \Omega$.

Noticing that e satisfies the error equation

$$a_h(e, v_h^*) = 0 \quad \forall v_h \in \mathring{S}_h(D), \tag{4.7}$$

we let $Z_h \in S_h(D)$ be a function that satisfies

$$a_h(Z_h, v_h^*) = 0 \quad \forall v_h \in \mathring{S}_h(D). \tag{4.8}$$

In fact, Z_h can be regarded as the discrete version of e in $S_h(D)$. The estimates of Z_h will play an important role in the local estimates of the FVE error e. Thus, in this subsection, we analyze Z_h in different norms. We begin with a negative-norm estimate for Z_h. The main idea is to combine the techniques established in [28] for the negative-norm estimates of the FEM and the estimates of the difference between the FEM and FVEM bilinear forms established in section 3.

LEMMA 4.1. *Let $D_0 = B(x_0, r_0) \subset D = B(x_0, r) \subset \Omega$ be two fixed concentric discs with $r_0, r > 0$ independent of h. Then for all integers $0 \le p \le k-1$ and for h sufficiently small, we have*

$$\|Z_h\|_{-p,D_0} \lesssim h^{1+p}\|Z_h\|_{1,D} + \|Z_h\|_{-p-1,D}. \tag{4.9}$$

Proof. Let $D' = B(x_0, \frac{2r_0+r}{3})$, $D'' = B(x_0, \frac{r_0+2r}{3})$, and $\omega \in C_0^\infty(D')$ with $\omega \equiv 1$ on D_0. Then for $p \ge 0$, we have

$$\|Z_h\|_{-p,D_0} \lesssim \|\omega Z_h\|_{-p,D'} = \sup_{g\in C_0^\infty(D')} \frac{(\omega Z_h, g)_{D'}}{\|g\|_{p,D'}} \lesssim \sup_{\psi\in C_0^\infty(D')} \frac{a(\omega Z_h, \psi)_{D'}}{\|\psi\|_{p+2,D'}}, \tag{4.10}$$

where ψ is the solution of (4.4). By (4.8), we have that for all $\chi \in \mathring{S}_h(D'')$,

$$\begin{aligned} a(\omega Z_h, \psi)_{D'} &= a(Z_h, \omega\psi) + (Z_h, \nabla\cdot(\nu\psi\nabla\omega)) + (Z_h, \nu\nabla\omega\cdot\nabla\psi) \\ &= a(Z_h, \omega\psi - \chi) + (Z_h, \nabla\cdot(\nu\psi\nabla\omega)) + (Z_h, \nu\nabla\omega\cdot\nabla\psi) + a(Z_h, \chi) \\ &= a(Z_h, \omega\psi - \chi) + (Z_h, \nabla\cdot(\nu\psi\nabla\omega)) + (Z_h, \nu\nabla\omega\cdot\nabla\psi) \\ &\quad + a(Z_h, \chi) - a_h(Z_h, \chi^*). \end{aligned} \tag{4.11}$$

Choosing χ as the nodal interpolation of $\omega\psi$ in $\mathring{S}_h(D'')$, we have

$$|a(\omega Z_h, \psi)|_{D'} \lesssim \|Z_h\|_{1,D'}\|\omega\psi - \chi\|_{1,D''} \lesssim h^{1+p}\|Z_h\|_{1,D}\|\psi\|_{p+2,D}. \tag{4.12}$$

Moreover, since $\omega \in C_0^\infty(D')$ and $\nu \in C^\infty(D')$, we have

$$|(Z_h, \nabla\cdot(\nu\psi\nabla\omega)) + (Z_h, \nu\nabla\omega\cdot\nabla\psi)| \lesssim \|Z_h\|_{-p-1,D}\|\psi\|_{p+2,D}. \tag{4.13}$$

On the other hand, by (4.2) and inverse inequality we have

$$\begin{aligned} |a(Z_h, \chi) - a_h(Z_h, \chi^*)| &\lesssim h^{1+p}\|Z_h\|_{1,D}\|\chi\|_{p+2,h,D} \\ &\lesssim h^{1+p}\|Z_h\|_{1,D}(\|\chi - \omega\psi\|_{p+2,h,D} + \|\omega\psi\|_{p+2,h,D}) \\ &\lesssim h^{1+p}\|Z_h\|_{1,D}\|\psi\|_{p+2,D}. \end{aligned} \tag{4.14}$$

Plugging (4.12)–(4.14) into (4.11), one obtains

$$a(\omega Z_h, \psi)_{D'} \lesssim (h^{1+p}\|Z_h\|_{1,D} + \|Z_h\|_{-p-1,D})\|\psi\|_{p+2,D}. \tag{4.15}$$

Combining (4.10) and (4.15), we obtain the estimate (4.9). ∎

LEMMA 4.2. *Given the conditions in Lemma* 4.1, *we obtain*

$$\|Z_h\|_{0,D_0} \lesssim h\|Z_h\|_{1,D} + \|Z_h\|_{-p-1,D}. \tag{4.16}$$

Proof. Let $D_0 \subset D_1 \subset \cdots \subset D_{p+1} = D$ be concentric discs with increasing radii. Setting $p = 0$ in (4.9), we have

$$\|Z_h\|_{0,D_0} \lesssim h\|Z_h\|_{1,D_1} + \|Z_h\|_{-1,D_1}. \tag{4.17}$$

Then we reapply (4.9) to estimate $\|Z_h\|_{-1,D_1}$. By $h \le 1$, we have

$$\|Z_h\|_{0,D_0} \lesssim h\|Z_h\|_{1,D_2} + \|Z_h\|_{-2,D_2}. \tag{4.18}$$

Continuing this process until to D_{p+1}, we obtain the desired result (4.16). ∎

LEMMA 4.3. *Given the conditions in Lemma* 4.1, *we have*

$$\|Z_h\|_{1,D_0} \lesssim h\|Z_h\|_{1,D} + \|Z_h\|_{-p-1,D}. \tag{4.19}$$

Proof. Let $D_0 \subset D' \subset D'' \subset D$ be fixed concentric discs with increasing radii and $\omega \in C_0^\infty(D')$ with $\omega \equiv 1$ on D_0. Then, we have

$$\|Z_h\|_{1,D_0} \le \|\omega Z_h\|_{1,D'} \le \|\omega Z_h - R_h(\omega Z_h)\|_{1,D''} + \|R_h(\omega Z_h)\|_{1,D''}, \tag{4.20}$$

where $R_h = R_h^{D''}$.

Next, we estimate the two terms on the right-hand side of (4.20). For the first term, we use (2.11), (4.1), and the inverse inequality to obtain

$$\begin{aligned}\|\omega Z_h - R_h(\omega Z_h)\|_{1,D''} &\lesssim \inf_{\zeta\in \mathring{S}_h(D'')} |\omega Z_h - \zeta|_{h,\frac{3}{2}+\varepsilon} \\ &= \inf_{\zeta\in \mathring{S}_h(D'')} \left(|\omega Z_h - \zeta|_{1,D''} + \sum_{\tau\in\mathcal{T}_h} h_\tau^{1/2+\varepsilon} |\omega Z_h - \zeta|_{\frac{3}{2}+\varepsilon,\tau} \right) \\ &\lesssim h\|Z_h\|_{1,D}, \end{aligned} \tag{4.21}$$

where ε is an arbitrary small positive constant. For the second term, letting $\phi = \frac{R_h(\omega Z_h)}{\|R_h(\omega Z_h)\|_{1,D''}}$, then $\|\phi\|_{1,D''} = 1$, $\phi \in \mathring{S}_h(D'')$ and we have

$$\|R_h(\omega Z_h)\|_{1,D''} \lesssim a_h(\omega Z_h, \phi^*)_{D''}. \tag{4.22}$$

By the definition of bilinear form a_h and (4.8), we have

$$\begin{aligned} a_h(\omega Z_h, \phi^*)_{D''} &= \sum_{E\in(\mathcal{E}_h'\cap D'')} [\phi^*]_E \int_E \nu \frac{\partial(\omega Z_h)}{\partial \mathbf{n}} ds \\ &= \sum_{E\in(\mathcal{E}_h'\cap D'')} \int_E \nu \frac{\partial Z_h}{\partial \mathbf{n}} [\omega\phi^* - \chi^*] ds + \sum_{E\in(\mathcal{E}_h'\cap D'')} [\phi^*]_E \int_E Z_h \nu \frac{\partial \omega}{\partial \mathbf{n}} ds \\ &\triangleq K_1 + K_2, \end{aligned} \tag{4.23}$$

where $\chi \in \mathring{S}_h(D'')$ is arbitrary. By the Cauchy–Schwartz inequality, the trace inequality, and the inverse inequality and the boundedness of ν, we have

$$\begin{aligned}|K_2| &\lesssim |\phi^*|'_{h,D''}\left(\sum_{E\in(\mathcal{E}'_h\cap D'')} h_E\int_E\left(Z_h\nu\frac{\partial\omega}{\partial\mathbf{n}}\right)^2 ds\right)^{\frac12}\\ &\lesssim |\phi^*|'_{h,D''}\left(\sum_{E\in(\mathcal{E}'_h\cap D'')}\sum_{E\cap\tau\neq\emptyset}(|Z_h\nabla\omega|^2_{0,\tau}+h_Q^2|Z_h\nabla\omega|^2_{1,\tau})\right)^{\frac12}\\ &\lesssim |\phi^*|'_{h,D''}\left(\sum_{\tau\in(\mathcal{T}_h\cap D'')}|Z_h|^2_{0,\tau}\right)^{\frac12}\lesssim|\phi|_{1,D''}\|Z_h\|_{0,D''}.\end{aligned}\tag{4.24}$$

To estimate K_1, we write

$$K_1 = K_{1,1}+K_{1,2}+K_{1,3}\tag{4.25}$$

with

$$\begin{aligned}K_{1,1} &= \sum_{E\in(\mathcal{E}'_h\cap D'')}\int_E \nu\frac{\partial Z_h}{\partial\mathbf{n}}[\omega\phi^*-\omega^*\phi^*]ds,\\ K_{1,2} &= \sum_{E\in(\mathcal{E}'_h\cap D'')}\int_E \nu\frac{\partial Z_h}{\partial\mathbf{n}}[\omega^*\phi^*-(I_h\omega)^*\phi^*]ds,\\ K_{1,3} &= \sum_{E\in(\mathcal{E}'_h\cap D'')}\int_E \nu\frac{\partial Z_h}{\partial\mathbf{n}}[(I_h\omega)^*\phi^*-\chi^*]ds.\end{aligned}$$

By the continuity of a_h and the boundedness of ν, we have

$$|K_{1,1}|\lesssim h|\omega|_{1,\infty}\sum_{E\in(\mathcal{E}'_h\cap D'')}\int_E\left|\frac{\partial Z_h}{\partial\mathbf{n}}[\phi^*]\right|ds\lesssim h|Z_h|_{1,D''}|\phi|_{1,D''}\tag{4.26}$$

and

$$|K_{1,2}|\lesssim\sum_{E\in(\mathcal{E}'_h\cap D'')}\left|\int_E\frac{\partial Z_h}{\partial\mathbf{n}}[(\omega-I_h\omega)^*][\phi^*]ds\right|\lesssim h|Z_h|_{1,D''}|\phi|_{1,D''}.\tag{4.27}$$

Moreover, following the proof of (4.24), we choose χ to satisfy (4.1) for ϕ. Then

$$|K_{1,3}|\lesssim|Z_h|_{1,D''}|(I_h\omega)\phi-\chi|_{1,D''}\lesssim h|Z_h|_{1,D''}|\phi|_{1,D''}.\tag{4.28}$$

Plugging (4.24)–(4.28) into (4.22) and combining with (4.21), it follows that

$$\|Z_h\|_{1,D_0}\lesssim h\|Z_h\|_{1,D''}+\|Z_h\|_{0,D''}.\tag{4.29}$$

For the term $\|Z_h\|_{0,D''}$ in (4.29), we can further apply the estimate (4.16) in Lemma 4.2 and D_0 replaced by D''. Then the estimate (4.19) follows from (4.20), (4.21), and (4.29). ∎

Using Lemma 4.3, we next show that the H^1 norm of Z_h in an interior domain can be bounded by its negative norm in a slightly larger interior region.

LEMMA 4.4. *Given the conditions in Lemma* 4.1, *we have*

$$\|Z_h\|_{1,D_0} \lesssim \|Z_h\|_{-p-1,D}. \tag{4.30}$$

Proof. Let $D_0 \subset D_1 \subset \cdots \subset D_{p+3} = D$ be concentric discs with increasing radii. Applying Lemma 4.3 with D_0 and D replaced by D_j, D_{j+1}, $0 \le j \le p+1$, we obtain

$$\|Z_h\|_{1,D_j} \lesssim h\|Z_h\|_{1,D_{j+1}} + \|Z_h\|_{-p-1,D_{j+1}}. \tag{4.31}$$

Stating with $j = 0$ and iterating $p+2$ times, one gets

$$\|Z_h\|_{1,D_0} \lesssim h^{p+2}\|Z_h\|_{1,D_{r+2}} + \|Z_h\|_{-p-1,D_{p+2}}. \tag{4.32}$$

From the inverse inequality, we have

$$h^{p+2}\|Z_h\|_{1,D_{p+2}} \lesssim \|Z_h\|_{-p-1,D}. \tag{4.33}$$

The inequality (4.30) then follows from (4.32) and (4.33). ∎

4.2. Interior estimates of the FVE error. In this section, we derive the interior H^1 and L^2 norm error estimates for the FVEM (2.3).

THEOREM 4.5. *Let $G_0 \subset G \subset \Omega$ be fixed interior subregions of Ω with $h \lesssim d :=$ dist$(\partial G_0, \partial G)$ and d be independent of h. Suppose $u \in H^{1+s}(G)$ ($\frac{1}{2} < s \le k$) and let the integer $0 \le p \le k-1$. Then we have*

$$\|u - u_h\|_{1,G_0} \lesssim h^s\|u\|_{s+1,G} + \|u - u_h\|_{-p,G}; \tag{4.34}$$

furthermore, if $u \in H^{s+2}(G)$ ($\frac{1}{2} < s \le k$), we have

$$\|u - u_h\|_{0,G_0} \lesssim h^{s+1}\|u\|_{s+2,G} + \|u - u_h\|_{-p,G}. \tag{4.35}$$

Proof. Using a covering argument, it suffices to show (4.34) and (4.35) for the case that G_0 and G are two concentric discs $D_0 \subset D \subset \Omega$ with increasing radii. In what follows, we let $D_0 \subset D_0' \subset D' \subset D'' \subset D$ be five fixed concentric discs with increasing radii, and the cut-off function $\omega \in C_0^\infty(D')$ satisfying $\omega \equiv 1$ on D_0'.

We first show (4.34). Letting $e = u - u_h$, for $R_h = R_h^{D''}$ we have

$$\|e\|_{1,D_0} \le \|\omega u - R_h(\omega u)\|_{1,D''} + \|R_h(\omega u) - u_h\|_{1,D_0}. \tag{4.36}$$

We observe that for any $v \in \mathring{S}_h(D_0')$,

$$a_h(R_h(\omega u) - u_h, v^*)_{D_0'} = a_h(\omega u - u_h, v^*)_{D_0'} = a_h(u - u_h, v^*)_{D_0'} = 0,$$

which implies (4.8) with $Z_h = R_h(\omega u) - u_h$ and D replaced by D_0'. Thus, by Lemma 4.4, we have

$$\begin{aligned} &\|R_h(\omega u) - u_h\|_{1,D_0} \\ &\quad\lesssim \|R_h(\omega u) - u_h\|_{-p,D_0'} \lesssim \|u - u_h\|_{-p,D_0'} + \|\omega u - R_h(\omega u)\|_{-p,D_0'} \\ &\quad\lesssim \|u - u_h\|_{-p,D} + \|\omega u - R_h(\omega u)\|_{1,D''} \lesssim \|u - u_h\|_{-p,D} + h^s\|u\|_{s+1,D}, \end{aligned} \tag{4.37}$$

where we have used (4.5) in the last inequality. On the other hand, by (4.5), we also have

$$\|\omega u - R_h(\omega u)\|_{1,D''} \lesssim h^s\|u\|_{s+1,D}. \tag{4.38}$$

Then by (4.36), (4.37), and (4.38), we obtain

$$\|e\|_{1,D_0} \lesssim h^s \|u\|_{s+1,D} + \|e\|_{-p,D}. \tag{4.39}$$

Thus, by the covering argument, the estimate (4.34) is obtained.

Next we show (4.35). Similar to (4.36), we have

$$\|e\|_{0,D_0} \le \|\omega u - R_h(\omega u)\|_{0,D''} + \|R_h(\omega u) - u_h\|_{0,D_0}. \tag{4.40}$$

By Lemma 4.2 with $Z_h = R_h(\omega u) - u_h$, we have

$$\|R_h(\omega u) - u_h\|_{0,D_0} \lesssim h\|R_h(\omega u) - u_h\|_{1,D_0'} + \|R_h(\omega u) - u_h\|_{-p,D_0'}. \tag{4.41}$$

Utilizing (4.37) with D_0 replaced by D_0', one gets

$$\|R_h(\omega u) - u_h\|_{1,D_0'} \lesssim \|e\|_{-p,D} + h^s\|u\|_{s+1,D}. \tag{4.42}$$

Moreover, by (4.6), we have

$$\begin{aligned} \|R_h(\omega u) - u_h\|_{-p,D_0'} &\lesssim \|\omega u - R_h(\omega u)\|_{-p,D_0'} + \|u - u_h\|_{-p,D_0'} \\ &\lesssim \|\omega u - R_h(\omega u)\|_{0,D''} + \|e\|_{-p,D} \lesssim h^{s+1}\|u\|_{s+2,D} + \|e\|_{-p,D}. \end{aligned} \tag{4.43}$$

Plugging (4.43) and (4.42) into (4.41), we obtain

$$\|R_h(\omega u) - u_h\|_{0,D_0} \lesssim h^{s+1}\|u\|_{s+2,D} + \|e\|_{-p,D}. \tag{4.44}$$

For the first term on the right-hand side of (4.40), using (4.6), we obtain

$$\|\omega u - R_h(\omega u)\|_{0,D''} \lesssim h^{s+1}\|u\|_{s+1,D}. \tag{4.45}$$

Then, plugging (4.42) and (4.44) into (4.40), we have

$$\|e\|_{0,D_0} \lesssim h^{s+1}\|u\|_{s+2,D} + \|e\|_{-p,D}. \tag{4.46}$$

The estimate (4.35) follows by the covering argument. ∎

Remark 4.6. The hidden constant in Theorem 4.5 depends on the domains G_0 and G that are arbitrary fixed. In practical computations, it is also important to quantify such dependence when $\mathrm{dist}(\partial G_0, \partial G)$ is close to h. This shall give rise to local estimates for the FV approximation in interior regions near the singular point. Note that for a disc $A := B(x_0, d) \subset \Omega$, the dilation $\hat{x} = (x - x_0)/d$ translates A into the unit disc $\hat{A} = B(0,1)$ and $S_h(A)$ into a new finite dimensional space $S_{h/d}(\hat{A})$. Using the scaling argument, the inequalities (4.34) and (4.35) are still valid for A with h replaced by h/d, where the hidden constant is independent of d.

Corollary 4.7. *Let $G_0 \subset G \subset \Omega$ be interior subregions of Ω and $u \in H^{1+s}(G)$ with $\frac{1}{2} < s \le k$ and $[s] = \max\{n \in \mathbb{Z} | n \le s\}$ ($\mathbb{Z}$ is the set of integers). Suppose $h \lesssim d := \mathrm{dist}(\partial G_0, \partial G) \lesssim 1$. Then, for $0 \le p \le k-1$, we have*

$$\|e\|_{1,G_0} \lesssim h^s|u|_{s+1,G} + (h/d)^s \sum_{j=0}^{[s]+1} d^{[s]-j}|u|_{[s]+1-j,G} + d^{-p-1}\|e\|_{-p,G}. \tag{4.47}$$

Furthermore, if $u \in H^{s+2}(G)$,

$$\|e\|_{0,G_0} \lesssim h^{s+1}|u|_{s+2,G} + (h/d)^{s+1} \sum_{j=0}^{[s]+2} d^{[s]+2-j}|u|_{[s]+2-j,G} + d^{-p}\|e\|_{-p,G}. \tag{4.48}$$

Proof. Let $D_0(x_i) \subset D(x_i) \subset \Omega$ be two concentric discs centered at x_i with radii $r/2$ and r, respectively. Note that $\bar{G}_0$ can be covered by a finite number of discs $D_0(x_i)$ such that $x_i \in \bar{G}_0$, and $\cup_i D(x_i)$ is a subset of G. Thus, it suffices to show the estimates (4.47) for the two discs $D_0(x_i)$ and $D(x_i)$. To simplify the notation, we let $D_0 = D_0(x_i)$ and $D = D(x_i)$ below.

We use a local coordinate system on D, such that x_i is the origin. Then, the dilation $\hat{x} = (x - x_i)/d$ translates D_0 and D to $\hat{D}_0$ and $\hat{D}$ with $\text{dist}(\partial\hat{D}_0, \partial\hat{D}) = 1/2$. Meanwhile, for a function v on D, we define $\hat{v}(\hat{x}) = v(x)$. Therefore, by the scaling argument, we have

$$|\hat{e}|_{\sigma,\hat{D}_0} = d^{\sigma-1}|e|_{\sigma,D_0}, \quad \sigma \geq 0, \tag{4.49}$$

and $\hat{e}$ satisfies

$$a_h(\hat{e}, \hat{v}^*) = 0, \tag{4.50}$$

where $\hat{v} \in \mathring{S}_{h/d}(\hat{D})$. Therefore, using Theorem 4.5 with $\hat{e}$ and h replaced by h/d on interior regions $\hat{D}_0 \subset \hat{D}$, we have

$$\|\hat{e}\|_{1,\hat{D}_0} \lesssim h^s d^{-s}\|\hat{u}\|_{s+1,\hat{D}} + \|\hat{e}\|_{-p,\hat{D}}, \tag{4.51}$$

$$\|\hat{e}\|_{0,\hat{D}_0} \lesssim h^{s+1} d^{-s-1}\|\hat{u}\|_{s+2,\hat{D}} + \|\hat{e}\|_{-p,\hat{D}}. \tag{4.52}$$

In addition, by the definition of the H^{-p} norm and $d \leq 1$, for $0 \leq j \leq p$, we have

$$\|\hat{e}\|_{-j,\hat{D}} \leq d^{-1-j}\|e\|_{-j,D}. \tag{4.53}$$

Then by (4.49) and (4.51)–(4.53), we obtain

$$\|e\|_{1,D_0} \lesssim h^s|u|_{s+1,D} + (h/d)^s \sum_{j=0}^{[s]+1} d^{[s]-j}|u|_{[s]+1-j,D} + d^{-p-1}\|e\|_{-p,D},$$

$$\|e\|_{0,D_0} \lesssim h^{s+1}|u|_{s+2,D} + (h/d)^{s+1} \sum_{j=0}^{[s]+2} d^{[s]+2-j}|u|_{[s]+2-j,D} + d^{-p}\|e\|_{-p,D},$$

which completes the proof. ∎

Remark 4.8. Theorem 4.5 is the special case of Corollary 4.7 where d is a constant independent of h. According to Theorem 4.5, the H^1 norm of the error in an interior region away from the singular point is bounded by the combination of the best local approximation error in the finite element space and a negative norm of the interior error. Note that for finite volume methods, the term $\|e\|_{-p,G}$ in general is determined by the smoothness of u and the adjoint problem.

Corollary 4.7 provides the interior error estimates when the boundary distance between the two interior regions is small. An implication of these estimates is that if the interior region of interest G_0 is close to the singular point, we have $d = \mathcal{O}(h)$. In this case, the additional factors (functions of d) in the estimates (4.47) can override the high-order convergence in $\|e\|_{-p,G}$ (see Theorem 3.2) and consequently make the upper bounds of the local error in G_0 comparable to the upper bounds of the global error.

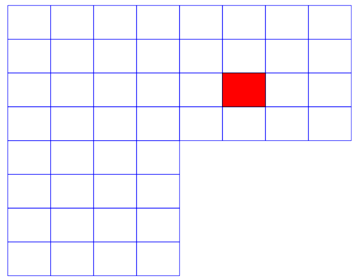

FIG. 5.1. *The computational domain of Example* 5.1 *and the interior domain (in red).*

5. Numerical tests. In this section, we provide numerical results to support our theoretical findings in Theorem 4.5.

Example 5.1. We consider the problem (1.1) with identity matrix on the domain $\Omega = ([-1,1]^2 \setminus [0,1] \times [-1,0])$ (i.e, L-shape Figure 5.1), which has the exact solution

$$u = a^{\frac{2}{3}} \sin\left(\frac{2}{3}\mu\right) - \frac{a^2}{4}, a = \sqrt{x^2+y^2}, \mu = \arctan\left(\frac{y}{x}\right),$$

and the right-hand function $f = 1$.

We see that $u \in H^{5/3-\epsilon}(\Omega)$, where $\epsilon > 0$ is arbitrarily small and hence $s = 2/3-\epsilon$. In this case, we see that $f \in C^\infty$. Since the largest interior angle of the domain is $\theta = \frac{3\pi}{2}$, we know the auxiliary problem associated with $\psi \in H^{1+t}$, where $0 < t < 2/3$. From (3.13), we derive that for any $0 \le p \le k-1$

$$\|e\|_{-p,\Omega} \lesssim h^{4/3-2\epsilon} \|u\|_{5/3-\epsilon,\Omega}. \tag{5.1}$$

For this specific example, the convergence order in any negative norm is no better than that in the $L^2(\Omega)$ norm. Meanwhile, we observe that u is singular at $z = (0,0)$, and u is of C^∞ in the interior domain $G_0 = [\frac{1}{4}, \frac{1}{4}]^2$ which is away from z.

Substituting the inequality (5.1) to estimates in Theorem 4.5, we obtain

$$\|e\|_{1,G_0} \lesssim h^k \|u\|_{k+1,G_1} + h^{4/3-2\epsilon} \|u\|_{5/3-\epsilon,\Omega}, \tag{5.2}$$

$$\|e\|_{0,G_0} \lesssim h^{k+1} \|u\|_{k+1,G_1} + h^{4/3-2\epsilon} \|u\|_{5/3-\epsilon,\Omega}, \tag{5.3}$$

where $G_1 \subset \Omega$ is slightly larger than G_0. Hence, for $k = 1$ the interior estimate in H^1 norm should be optimal, but for $k \ge 2$, it should be at most $h^{4/3-2\epsilon}$. While for the L^2 norm, the interior estimate yields no better than the global convergence.

In Table 5.1, we display the numerical results for the finite volume approximation of Example 5.1. We see in the table that the estimates in the global L^2 and H^1 norms are $\mathcal{O}(h^{4/3-2\epsilon})$ and $\mathcal{O}(h^{2/3-\epsilon})$ for both $k = 1, 2$, while the local error in an interior domain G_0 is optimal only for $k = 1$ in the H^1 norm but for other cases the convergent rates are around $h^{\frac{4}{3}}$. It demonstrates that the numerical results are consistent with our theoretical prediction in (5.2) and (5.3) and therefore verify the theory.

Example 5.2. We consider the problem on the domain $\Omega = [0,1]^2$ with the exact solution

$$u = x(1-x)y(1-y)a^{-\frac{3}{2}},$$

where $a = \sqrt{x^2+y^2}$.

Table 5.1
Example 5.1: *The convergence rates.*

	h	Global domain Ω $\|e\|_{0,\Omega}$	Order	$\|e\|_{1,\Omega}$	Order	Interior domain G_0 $\|e\|_{0,G_0}$	Order	$\|e\|_{1,G_0}$	Order
Q^1	1/64	7.40E-04	–	3.53E-02	–	6.54E-05	–	1.49E-03	–
	1/128	2.93E-04	1.34	2.23E-02	0.66	2.57E-05	1.35	7.40E-04	1.01
	1/256	1.16E-04	1.34	1.44E-02	0.66	1.01E-05	1.34	3.69E-04	1.00
	1/512	4.58E-05	1.34	9.11E-03	0.66	4.00E-06	1.34	1.84E-04	1.00
Q^2	1/64	1.39E-04	–	1.54E-02	–	1.16E-05	–	4.57E-05	–
	1/128	5.45E-05	1.35	9.72E-03	0.67	4.61E-06	1.33	1.77E-05	1.37
	1/256	2.15E-05	1.34	6.13E-03	0.67	1.83E-06	1.33	6.93E-06	1.35
	1/512	8.47E-06	1.34	3.86E-03	0.67	7.25E-07	1.33	2.74E-06	1.34

Table 5.2
Example 5.2: *The convergence rates.*

	h	Global domain Ω $\|e\|_{0,\Omega}$	Order	$\|e\|_{1,\Omega}$	Order	Interior domain G_0 $\|e\|_{0,G_0}$	Order	$\|e\|_{1,G_0}$	Order
Q^2	1/8	2.03E-03	–	1.46E-01	–	1.31E-05	–	3.94E-04	–
	1/16	7.23E-04	1.49	1.04E-01	0.49	2.39E-06	2.45	9.89E-05	1.99
	1/32	2.57E-04	1.49	7.38E-02	0.49	4.25E-07	2.50	2.47E-05	2.00
	1/64	9.10E-05	1.50	5.23E-02	0.50	7.48E-08	2.50	6.18E-06	2.00
Q^3	1/32	1.07E-04	–	4.67E-02	–	7.11E-08	–	3.94E-07	–
	1/64	3.79E-05	1.50	3.30E-02	0.50	1.26E-08	2.50	6.14E-08	2.67
	1/128	1.34E-05	1.50	2.33E-02	0.50	2.23E-09	2.50	1.01E-08	2.61
	1/256	4.74E-06	1.50	1.65E-02	0.50	3.94E-10	2.50	1.70E-09	2.56

We can see that $u \in H^{3/2-\epsilon}(\Omega)$ for $\epsilon > 0$ arbitrarily small and hence $s = 1/2 - \epsilon$. Note that the singularity in u is around the origin $z = (0, 0)$. The largest interior angle is $\theta = \pi/2$, which indicates that the function in the duality argument $\psi \in H^{1+t}(\Omega)$ with $0 < t < 2$. From (3.13), for any $1 \leq p \leq k - 1$ with $k \geq 2$ we can derive

$$\|e\|_{-p} \lesssim h^{2.5-2\epsilon}\|u\|_{3/2-\epsilon}. \tag{5.4}$$

By (2.14) and (2.10) we expect to see the convergence in the global L^2 and H^1 norms be $\mathcal{O}(h^{3/2-\epsilon})$ and $\mathcal{O}(h^{1/2-\epsilon})$, respectively. Considering the interior domain $G_0 = [\frac{1}{2}, \frac{3}{4}]^2$, the solution u is smooth. Applying (5.4) in Theorem 4.5, we obtain that for all $u \in H^{k+1}(G_1)$ (G_1 is slightly larger than G_0 and u is smooth in G_1)

$$\|e\|_{1,G_0} \lesssim h^k\|u\|_{k+1,G_1} + h^{2.5-2\epsilon}\|u\|_{5/3-\epsilon,\Omega}, \tag{5.5}$$

$$\|e\|_{0,G_0} \lesssim h^{k+1}\|u\|_{k+1,G_1} + h^{2.5-2\epsilon}\|u\|_{5/3-\epsilon,\Omega}. \tag{5.6}$$

Therefore, for $k = 2$ the interior convergence rates should be $\mathcal{O}(h^2)$ (resp., $\mathcal{O}(h^{2.5-2\epsilon})$) in H^1 (resp., L^2) norm, while the interior accuracy should be $\mathcal{O}(h^{2.5-2\epsilon})$ in H^1 and L^2 norms for $k = 3$.

The numerical results of Example 5.2 are listed in Table 5.2. We can see from the table that the interior and global estimates are consistent with our theoretical prediction.

6. Conclusion. The error analysis for high-order FVEM is a challenging task. This paper is one in a series that attempts to set up a mathematical foundation for a family of high-order FVEM over quadrilateral meshes. In previous works [17, 25, 43], we analyzed the stability, H^1 error, L^2 error, and maximum-norm error of high order

FVEM over quadrilateral meshes. In this article, we present our study on negative-norm error estimates and interior error estimates, especially for problems with low-regularity solutions.

We point out that as in the L^2-norm estimates for the FVEM, we require a slightly stronger regularity to achieve optimal convergence order for the negative-norm error estimates than that for the FEM. Consequently, the regularity requirement to obtain optimal convergence order in a local domain is also a little bit stronger than that for the FEM.

REFERENCES

[1] R. E. Bank and D. J. Rose, *Some error estimates for the box method*, SIAM J. Numer. Anal., 24 (1987), pp. 777–787.

[2] T. Barth and M. Ohlberger, *Finite Volume Methods: Foundation and Analysis*, Encyclopedia Comput. Mech., Wiley, New York, 2004.

[3] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 2008.

[4] Z. Cai, *On the finite volume element method*, Numer. Math., 58 (1991), pp. 713–735.

[5] Z. Cai, J. Douglas, and M. Park, *Development and analysis of higher order finite volume methods over rectangles for elliptic equations*, Adv. Comput. Math., 19 (2003), pp. 3–33.

[6] S. H. Chou and D. Y. Kwak, *Multigrid algorithms for a vertex-centered covolume method for elliptic problems*, Numer. Math., 90 (2002), pp. 441–458.

[7] P. Chatzipantelidis, *A finite volume method based on the Crouzeix-Raviart element for elliptic PDEs in two dimensions*, Numer. Math., 82 (1999), pp. 409–432.

[8] L. Chen, *A new class of high order finite volume methods for second order elliptic equations*, SIAM J. Numer. Anal., 47 (2010), pp. 4021–4043.

[9] Z. Chen, J. Wu and Y. Xu, *Higher-order finite volume methods for elliptic boundary value problems*, Adv. Comput. Math., 37 (2013), pp. 191–253.

[10] Ph. Emonot, *Methods de Volums Elements Finis: Applications aux Equations de Navier-Stokes et Resultats de Convergence*, Ph.D. thesis, University of Lyon, France, 1992.

[11] R. E. Ewing, T. Lin, and Y. Lin, *On the accuracy of finite volume element method based on piecewise linear polynomials*, SIAM J. Numer. Anal., 39 (2002), pp. 1865–1888.

[12] R. Eymard, T. Gallouet, and R. Herbin, *Finite Volume Methods*, in Handbook of Numerical Analysis VII, North-Holland, Amsterdam, 2000, pp. 713–1020.

[13] S. Faure, J. Laminie, and R. Teman, *Finite Volume Discretization and Multilevel Methods in Flow Problems*, J. Sci. Comput., 25 (2005), pp. 231–261.

[14] P. Grisvard,*Singularities in Boundary Value Problems*, Springer, New York, 1992.

[15] L. Guo, H. Li, and Y. Yang, *Interior estimates of semidiscrete finite element methods for parabolic problems with distributional data*, J. Comput. Math., 37 (2019), pp. 460–476.

[16] W. Hackbusch, *On first and second order box methods*, Computing, 41 (1989), pp. 277–296.

[17] W. He, Z. Zhang, and Q. Zou, *Maximum-norms error estimates for high-order finite volume schemes over quadrilateral meshes*, Numer. Math., 138 (2017), pp. 473–500.

[18] J. G. Huang and S. Xi, *On the finite volume element method for general self-adjoint elliptic problems*, SIAM J. Numer. Anal., 35 (1998), pp. 1762–1774.

[19] J. M. Hyman, R. Knapp, and J. C. Scovel, *High order finite volume approximations of differential operators on nonuniform grids*, Phys. D, 60 (1992), pp. 112–138.

[20] R. D. Lazarov, I. D. Michev, and P. S. Vassilevski, *Finite volume methods for convection-diffusion problems*, SIAM J. Numer. Anal., 33 (1996), pp. 31–55.

[21] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 2002.

[22] H. Li, L. Mu, and X. Ye, *Interior energy error estimates for the weak Galerkin finite element method*, Numer. Math., 139 (2018), pp. 447–478.

[23] R. Li, Z. Chen, and W. Wu, *The Generalized Difference Methods for Partial Differential Equations*, Numer. Anal. Finite Vol. Methods, Marcel Dekker, New York, 2000.

[24] F. Liebau, *The finite volume element method with quadratic basis function*, Computing, 57 (1996), pp. 281–299.

[25] Y. Lin, M. Yang, and Q. Zou, *L^2 error estimates for a class of any order finite volume schemes over quadrilateral meshes*, SIAM J. Numer. Anal., 53 (2015), pp. 2030–2050.

[26] J. Lv and Y. Li, *Optimal biquadratic finite volume element methods on quadrilateral meshes*, SIAM J. Numer. Anal., 50 (2013), pp. 2397–2399.

[27] R. A. Nicolaides, T. A. Porsching, and C. A. Hall, *Covolume methods in computational fluid dynamics*, in Computational Fluid Dynamics Review, M. Hafez and K. Oshima, eds., Wiley, New York, 1995, pp. 279–299.

[28] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp., 28 (1974), pp. 937–958.

[29] C. Ollivier-Gooch and M. Altena, *A high-order-accurate unconstructed mesh finite-volume scheme for the advection-diffusion equation*, J. Comput. Phys., 181 (2002), pp. 729–752.

[30] S. V. Patankar, *Numerical Heat Transfer and Fluid Flow*, Ser. Comput. Methods Mech. Thermal Sci., McGraw Hill, New York, 1980.

[31] M. Plexousakis and G. E. Zouraris, *On the construction and analysis of high order locally conservative finite volume-type methods for one-dimensional elliptic problems*, SIAM J. Numer. Anal., 42 (2004), pp. 1226–1260.

[32] A. Quateroni, R. Sacco, and F. Saleri, *Numerical Mathematics*, Springer, New York, 2007.

[33] A. H. Schatz and L. B. Wahlbin, *Interior maximum norm estimates for finite element methods*, Math. Comp., 31 (1977), pp. 414–442.

[34] A. H. Schatz and L. B. Wahlbin, *Maximum norm estimates in the finite element method on plane polygonal domains. Part* I, Math. Comp., 32 (1978), pp. 73–109.

[35] A. H. Schatz and L. B. Wahlbin, *On the quasi-optimality in L^∞ of the H^1-projection into finite element spaces*, Math. Comp., 38 (1982), pp. 1–22.

[36] T. Schmidt, *Box schemes on quadrilateral meshes*, Computing, 51 (1993), pp. 271–292.

[37] C. Shu, *High order finite difference and finite volume WENO schemes and discontinuous Galerkin methods for CFD*, J. Comput. Fluid Dyn., 17 (2003), pp. 107–118.

[38] E. Süli, *The accuracy of cell vertex finite volume methods on quadrilateral meshes*, Math. Comp., 59 (1992), pp. 359–382.

[39] P. Sun, G. Xue, C. Wang, and J. Xu, *A combined finite element-upwind finite volume-Newton's method for liquid-feed direct methanol fuel cell simulations*, in Proceedings of the Sixth International Fuel Cell Science, Engineering and Technology Conference 8, Denver, CO, 2008.

[40] M. Tian and Z. Chen, *Quadratic element generalized differential methods for elliptic equations*, Numer. Math. J. Chinese Univ., 13 (1991), pp. 99–113.

[41] X. Wang and Y. Li, *L^2 Error estimates for high order finite volume methods on triangular meshes*, SIAM J. Numer. Anal., 54 (2016), pp. 2729–2749.

[42] J. Xu and Q. Zou, *Analysis of linear and quadratic simplitical finite volume methods for elliptic equations*, Numer. Math., 111 (2009), pp. 469–492.

[43] Z. Zhang and Q. Zou, *Vertex-centered finite volume schemes of any order over quadrilateral meshes for elliptic boundary value problems*, Numer. Math., 130 (2015), pp. 363–393.