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# The macroelement analysis for axisymmetric Stokes equations ☆

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ABSTRACT

We consider the mixed finite element approximation of the axisymmetric Stokes problem (ASP) on a bounded polygonal domain in the $rz$-plane. Standard stability results on mixed methods do not apply due to the singular coefficients in the differential operator and due to the singular or vanishing weights in the associated function spaces. We develop new finite element analysis in these weighted spaces, and propose macroelement conditions that are sufficient to ensure the well-posedness of the mixed methods for the ASP. These conditions are local, relatively easy to verify, and therefore will be useful for validating the stability of a variety of mixed finite element methods. These new conditions can not only re-verify existing stable mixed methods for the ASP, but also lead to the discovery of new stable conservative mixed methods. We report numerical test results that confirm the theory.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^2_+ = \{(r,z),\ r>0\}$ be a bounded polygonal domain in the $rz$-plane. Let $\Gamma_0$ be the interior part of $\partial\Omega \cap \{r=0\}$ and $\Gamma := \partial\Omega \setminus \Gamma_0$. Given $f_r$ and $f_z$, we consider the following axisymmetric Stokes problem (ASP): Find $u_r$ and $u_z$ satisfying,

$$\begin{cases} -(\partial_r^2 + r^{-1}\partial_r + \partial_z^2 - r^{-2})u_r + \partial_r p = f_r & \text{in} \quad \Omega \\ -(\partial_r^2 + r^{-1}\partial_r + \partial_z^2)u_z + \partial_z p = f_z & \text{in} \quad \Omega \\ (\partial_r + r^{-1})u_r + \partial_z u_z = 0 & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

with the boundary conditions

$$u_z = 0 \text{ on } \Gamma, \quad \partial_r u_z = 0 \text{ on } \Gamma_0, \quad \text{and} \quad u_r = 0 \text{ on } \partial\Omega.$$

Equation (1.1) is important in studying the three-dimensional (3D) Stokes problem on axisymmetric domains.

For instance, denote by $(r,\theta,z)$ (resp. $(x,y,z)$) the cylindrical (resp. Cartesian) coordinates of a point in $\mathbb{R}^3$. Let $\widetilde{\Omega} := (\Omega \cup \Gamma_0) \times [-\pi,\pi) \subset \mathbb{R}^3$ be the domain formed by the rotation of $\Omega$ about the $z$-axis (Fig. 1). A 3D vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ (resp. function $v$) is axisymmetric if

$$\mathcal{R}_{-\sigma}(\boldsymbol{v}\circ\mathcal{R}_\sigma) = \boldsymbol{v} \qquad (\text{resp. } v\circ\mathcal{R}_\sigma(x,y,z) = v(x,y,z)), \quad \forall (x,y,z) \in \widetilde{\Omega}, \tag{1.2}$$

where

$$\mathcal{R}_\sigma = \begin{pmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

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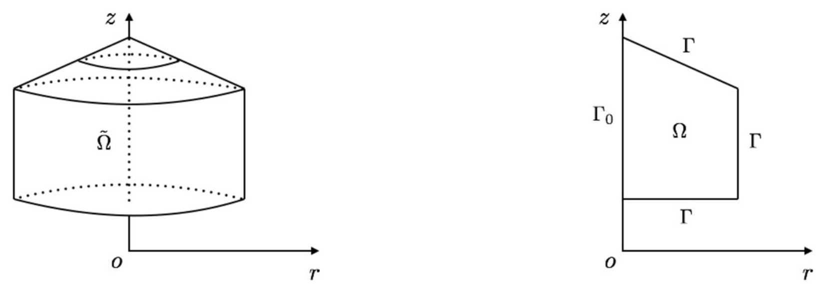


**Fig. 1.** An axisymmetric domain $\widetilde{\Omega}$ (left) and the meridian domain $\Omega$ (right).

is the rotation about the $z$-axis with angle $\sigma$. In addition, a 3D vector field can also be expressed by its cylindrical (radial, angular, and axial) components

$$\boldsymbol{v} = (v_r, v_\theta, v_z) = (v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta, v_3).$$

It can be shown that if $\boldsymbol{v}$ is axisymmetric, its cylindrical components are all axisymmetric functions (Proposition 2.2). For an axisymmetric function $v(r,\theta,z)$, since it is invariant under rotation, we let $v(r,z) := v(r,\theta,z)$ be its trace on the meridian domain $\Omega$. Thus, when the vector fields and functions involved are axisymmetric, the 3D Stokes problem

$$\begin{cases} -\Delta\widetilde{\boldsymbol{u}} + \nabla\widetilde{p} = \widetilde{\boldsymbol{f}} & \text{in} \quad \widetilde{\Omega} \\ \operatorname{div}\widetilde{\boldsymbol{u}} = 0 & \text{in} \quad \widetilde{\Omega} \\ \widetilde{\boldsymbol{u}} = 0 & \text{on} \quad \partial\widetilde{\Omega} \end{cases} \tag{1.4}$$

can be reduced to a system of two decoupled equations: the ASP (1.1) and the scalar azimuthal Stokes equation

$$\begin{cases} -(\partial_r^2 + r^{-1}\partial_r + \partial_z^2 - r^{-2})u_\theta = f_\theta & \text{in} \quad \Omega \\ u_\theta = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.5}$$

where $(u_r, u_\theta, u_z, p)$ and $(f_r, f_\theta, f_z)$ are the traces on $\Omega$ of the axisymmetric data $(\widetilde{\boldsymbol{u}}, \widetilde{p}) = (\widetilde{u}_r, \widetilde{u}_\theta, \widetilde{u}_z, \widetilde{p})$ and $\widetilde{\boldsymbol{f}} = (\widetilde{f}_r, \widetilde{f}_\theta, \widetilde{f}_z)$ in the 3D problem (1.4).

We are interested in mixed finite element approximations of the ASP (1.1). For numerical analysis of equation (1.5), we refer to [6,22] and references cited therein. Note that for the 3D problem (1.4) with general data, the $k$th Fourier coefficients of the pressure and two velocity components (radial and axial) are determined by equations similar to (1.1). Therefore, the study of the ASP shall also shed light on further numerical advances for 3D Stokes equations on axisymmetric domains with general data.

The dimensional reduction from 3D to 2D ((1.4) $\to$ (1.1)+(1.5)) can significantly reduce the computational cost solving the Stokes problem in 3D axisymmetric domains. However, the coordinate transformation (Cartesian $\to$ cylindrical) leads to new differential operators with singular coefficients and new function spaces with singular or vanishing weights. In particular, a mixed finite element formulation for the ASP results in an indefinite discrete system. Rigorous numerical analysis is necessary to develop well-posed mixed methods in these weighted spaces. Using different analytical tools, there have been recent works in this direction. For example, the $P_1\text{iso}P_2 - P_1$ element was analyzed and proven to be stable for the ASP in [5]; the lowest-order Hood-Taylor element was studied in [14]; in [19,20], the stability property of general Hood-Taylor elements was established. In addition, the development of effective numerical methods for other axisymmetric problems has also drawn a lot of attention from the computational community, for which we mention the following relevant results. A detailed discussion on spectral methods for various axisymmetric problems can be found in [6]; for finite element approximations of the axisymmetric Poisson equation, we mention [17,21,22,24,25]; for numerical analysis of axisymmetric Maxwell equations, we refer to [3,16]; and the axisymmetric Stokes-Darcy flow was studied in [1,13].

In this paper, we develop a unified macroelement stability analysis framework for mixed methods solving the ASP, and in turn obtain local conditions on the finite element space that ensure the global well-posedness of the numerical scheme. To be more specific, let $b(\cdot,\cdot)_M$ and $b^s(\cdot,\cdot)_M$ be the bilinear forms defined by the axisymmetric divergence operator and by the usual divergence operator, respectively, on the local patch (macroelement) of the mesh. Then, if the associated null-spaces (4.3) consist of only the constant function in the local discrete pressure space, the mixed finite element method shall satisfy the inf-sup condition (Theorem 4.9). Namely, our result generalizes Stenberg's macroelement analysis for the usual Stokes equations [26] to the ASP. Therefore, it can be used to validate and design stable mixed finite element methods for equation (1.1).

Our analysis is technical, since various new estimates are needed for the unconventional divergence operator in weighted Sobolev spaces. This gives rise to a notable difference between our macroelement condition for the ASP and the macroelement condition for the usual Stokes equations in [26] and [27]. Namely, we require conditions for two bilinear forms $b(\cdot,\cdot)_M$ and $b^s(\cdot,\cdot)_M$, while only the condition for $b^s(\cdot,\cdot)_M$ was needed in [26] and [27]. Meanwhile, compared with the results in [5,14,19], where specific mixed formulations for the ASP were studied on triangular meshes, we propose stability conditions that apply to more general mixed finite element methods, on both triangular and quadrilateral meshes. To demonstrate applications of such conditions, we propose new locally conservative mixed finite element methods for the ASP. This is the first time these mixed methods are shown to be stable for the ASP. Numerical test results will be reported to verify the new findings. In addition, some of the estimates in this paper shall be useful for the analysis of other axisymmetric problems (e.g., axisymmetric linear elasticity and Maxwell equations) that are defined in similar weighted spaces.

The rest of the paper is organized as follows. In Section 2, we first introduce the appropriate weighed space for the ASP and its connections with the usual 3D Sobolev space. Then, we formulate mixed finite element methods based on the variational formation in weighted spaces. In Section 3, we define an interpolation operator onto the discrete velocity space that is exact in the sense of (3.6). Through a series of intermediate estimates, we further obtain the stability of this interpolation operator (Theorem 3.10). In Section 4, after providing the definition of the macroelement (Definition 4.1) and the macroelement condition (Assumption 4.2), we develop necessary tools for the macroelement analysis. We summarize the main result in Theorem 4.9. Namely, the macroelement condition is sufficient for the well-posedness of the mixed method solving the ASP. We also provide stable mixed methods that can be verified by the macroelement condition as well as some new locally conservative stable elements for the ASP. Numerical experiments are provided in Section 5. Concluding remarks are given in Section 6.

Throughout the paper, we adopt the bold notation for vector fields. Tildes are used for either axisymmetric vector fields or axisymmetric scalar functions in the axisymmetric domain $\widetilde{\Omega}$. Given a domain, we use the standard notation $H^m$ for the Sobolev spaces consisting of functions whose $k$th derivatives $(0 \le k \le m)$ are square integrable, and $L^2 := H^0$. By $a \simeq b$, we mean that there are constants $C_1, C_2 > 0$, independent of the individual element and sub-domain, such that $C_1 b \le a \le C_2 b$. The generic constant $C > 0$ in our analysis below may be different at different occurrences. It will depend on the computational domain, but not on the functions involved in the estimates or the mesh size in the finite element algorithms. For a 3D vector field $\boldsymbol{v}$, we refer to $(v_1, v_2, v_3)$ as its Cartesian components, and $(v_r, v_\theta, v_z)$ as its cylindrical components.

## 2. Preliminaries and notation

In this section, we introduce the notation and some preliminary estimates needed for further analysis.

### 2.1. Weighted spaces and the weak formulation

We begin with the definition of a family of weighted spaces [6] for the ASP (1.1).

**Definition 2.1.** (Weighted Sobolev Spaces). Recall the 2D meridian domain $\Omega$. For an integer $m \ge 0$, define

$$L^2_1(\Omega) := \left\{ v, \int_\Omega v^2 r\,dr\,dz < \infty \right\}, \quad H^m_1(\Omega) := \left\{ v,\ \partial^\alpha_c v \in L^2_1(\Omega),\ |\alpha| \le m \right\},$$

where the multi-index $\alpha = (\alpha_1, \alpha_2)$ is a pair of nonnegative integers, $|\alpha| := \alpha_1 + \alpha_2$, and $\partial^\alpha_c := \partial^{\alpha_1}_r \partial^{\alpha_2}_z$. With $H^0_1(\Omega) = L^2_1(\Omega)$, the norms and the semi-norms for any $v \in H^m_1(\Omega)$ are

$$\|v\|^2_{H^m_1(\Omega)} := \sum_{|\alpha| \le m} \int_\Omega (\partial^\alpha_c v)^2 r\,dr\,dz, \quad |v|^2_{H^m_1(\Omega)} := \sum_{|\alpha| = m} \int_\Omega (\partial^\alpha_c v)^2 r\,dr\,dz.$$

In addition, we define two more spaces $H^m_+(\Omega)$ and $H^m_-(\Omega)$ as follows.

For $H^m_+(\Omega)$, if $m$ is not even,

$$H^m_+(\Omega) := \left\{ v \in H^m_1(\Omega),\ \partial^{2i-1}_r v|_{\{r=0\}} = 0,\ 1 \le i < \frac{m}{2} \right\} \quad \text{with} \quad \|v\|_{H^m_+(\Omega)} = \|v\|_{H^m_1(\Omega)}. \tag{2.1}$$

If $m$ is even, besides the condition in (2.1), we require $\int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr\,dz < \infty$ for any $v \in H^m_+(\Omega)$, and the corresponding norm is

$$\|v\|_{H^m_+(\Omega)} = \left( \|v\|^2_{H^m_1(\Omega)} + \int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr\,dz \right)^{1/2}. \tag{2.2}$$

For $H^m_-(\Omega)$, if $m$ is not odd,

$$H^m_-(\Omega) := \left\{ v \in H^m_1(\Omega),\ \partial^{2i}_r v|_{\{r=0\}} = 0,\ 0 \le i < \frac{m-1}{2} \right\} \quad \text{with} \quad \|v\|_{H^m_-(\Omega)} = \|v\|_{H^m_1(\Omega)}. \tag{2.3}$$

If $m$ is odd, besides the condition in (2.3), we require $\int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr\,dz < \infty$, for any $v \in H^m_-(\Omega)$, and the corresponding norm is

$$\|v\|_{H^m_-(\Omega)} = \left( \|v\|^2_{H^m_1(\Omega)} + \int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr\,dz \right)^{1/2}.$$

We will also need the following subspaces:

$$H^1_{1,0}(\Omega) := H^1_1(\Omega) \cap \{v|_\Gamma = 0\}, \quad H^1_{-,0}(\Omega) := H^1_-(\Omega) \cap \{v|_{\partial\Omega} = 0\},$$

$$H^1_{+,0}(\Omega) := H^1_+(\Omega) \cap \{v|_\Gamma = 0\}, \quad L^2_{1,0}(\Omega) := L^2_1(\Omega) \cap \left\{ v : \int_\Omega v r\,dr\,dz = 0 \right\}.$$

Let $\widetilde{\mathbf{H}}^m(\widetilde{\Omega}) \subset [H^m(\widetilde{\Omega})]^3$ (resp. $\widetilde{H}^m(\widetilde{\Omega}) \subset H^m(\widetilde{\Omega})$) be the subspace consisting of axisymmetric vector fields (resp. functions). Then, the connections between different spaces on $\widetilde{\Omega}$ and on $\Omega$ can be summarized as follows.

**Proposition 2.2.** *For $\widetilde{v} \in \widetilde{H}^m(\widetilde{\Omega})$, define $v(r,z) = \widetilde{v}(r,\theta,z)$. Then, the mapping $\widetilde{v}(r,\theta,z) \to v(r,z)$ defines an isomorphism*

$$\widetilde{H}^m(\widetilde{\Omega}) \to H^m_+(\Omega). \tag{2.4}$$

*For $\widetilde{\mathbf{v}} \in \widetilde{\mathbf{H}}^m(\widetilde{\Omega})$, all its cylindrical components ($\widetilde{v}_r, \widetilde{v}_\theta$, and $\widetilde{v}_z$) are axisymmetric functions. Let*

$$v_r(r,z) := \widetilde{v}_r(r,\theta,z), \quad v_\theta(r,z) := \widetilde{v}_\theta(r,\theta,z), \quad v_z(r,z) := \widetilde{v}_z(r,\theta,z). \tag{2.5}$$

*Then, the mapping* $\widetilde{\boldsymbol{v}} \to (v_r, v_\theta, v_z)$ *defines an isomorphism*

$$\widetilde{\mathbf{H}}^m(\widetilde{\Omega}) \to H_-^m(\Omega) \times H_-^m(\Omega) \times H_+^m(\Omega). \tag{2.6}$$

*In addition, let* $\widetilde{\mathbf{H}}_0^1(\widetilde{\Omega}) \subset \widetilde{\mathbf{H}}^1(\widetilde{\Omega})$ *be the subspace with zero trace on the boundary* $\partial\widetilde{\Omega}$*. Recall the functions* $v_r, v_\theta$*, and* $v_z$ *from (2.5). Then, the mapping* $\widetilde{\boldsymbol{v}} \to (v_r, v_\theta, v_z)$ *defines an isomorphism*

$$\widetilde{\mathbf{H}}_0^1(\widetilde{\Omega}) \to H_{-,0}^1(\Omega) \times H_{-,0}^1(\Omega) \times H_{+,0}^1(\Omega). \tag{2.7}$$

**Proof.** The isomorphic mappings (2.4) and (2.6) are given in Theorem II.2.1 and Theorem II.2.6 from [6]. Therefore, we proceed to show (2.7).

Recall the following scalar version of (2.7) in II.4 from [6]: For $\widetilde{v} \in \widetilde{H}_0^1(\widetilde{\Omega}) := \widetilde{H}^1(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$, define $v(r,z) = \widetilde{v}(r,\theta,z)$; then, the mapping $\widetilde{v}(r,\theta,z) \to v(r,z)$ defines an isomorphism

$$\widetilde{H}_0^1(\widetilde{\Omega}) \to H_{+,0}^1(\Omega). \tag{2.8}$$

Then, for $(v_r, v_\theta, v_z) \in H_{-,0}^1(\Omega) \times H_{-,0}^1(\Omega) \times H_{+,0}^1(\Omega) \subset H_-^1(\Omega) \times H_-^1(\Omega) \times H_+^1(\Omega)$, let the axisymmetric functions $\widetilde{v}_r(r,\theta,z) = v_r(r,z)$, $\widetilde{v}_\theta(r,\theta,z) = v_\theta(r,z)$, and $\widetilde{v}_z(r,\theta,z) = v_z(r,z)$ be the cylindrical components of a vector field $\widetilde{\boldsymbol{v}}$. By the isomorphic mapping (2.6), $\widetilde{\boldsymbol{v}} \in \widetilde{\mathbf{H}}^1(\widetilde{\Omega})$. Note that its Cartesian components can be written as

$$v_1 = \widetilde{v}_r \cos\theta - \widetilde{v}_\theta \sin\theta, \quad v_2 = \widetilde{v}_r \sin\theta + \widetilde{v}_\theta \cos\theta, \quad v_3 = \widetilde{v}_z.$$

Since $\widetilde{v}_r|_{\partial\widetilde{\Omega}} = \widetilde{v}_\theta|_{\partial\widetilde{\Omega}} = \widetilde{v}_z|_{\partial\widetilde{\Omega}} = 0$ by the definition, $v_1$, $v_2$, and $v_3$ are also zero on $\partial\widetilde{\Omega}$. Therefore, $\widetilde{\boldsymbol{v}} \in \widetilde{\mathbf{H}}_0^1(\widetilde{\Omega})$.

Conversely, for $\widetilde{v} \in \widetilde{\mathbf{H}}_0^1(\widetilde{\Omega}) \subset \widetilde{\mathbf{H}}^1(\widetilde{\Omega})$, by (2.6), the mapping $\widetilde{\boldsymbol{v}} \to (v_r, v_\theta, v_z)$ defines an isomorphism

$$\widetilde{\mathbf{H}}^1(\widetilde{\Omega}) \to H_-^1(\Omega) \times H_-^1(\Omega) \times H_+^1(\Omega). \tag{2.9}$$

Since the Cartesian components $v_1, v_2, v_3 \in H_0^1(\widetilde{\Omega})$, its cylindrical components

$$\widetilde{v}_r = v_1 \cos\theta + v_2 \sin\theta, \quad \widetilde{v}_\theta = -v_1 \sin\theta + v_2 \cos\theta, \quad \widetilde{v}_z = v_3,$$

are also zero on $\partial\widetilde{\Omega}$. Therefore, $\widetilde{v}_r, \widetilde{v}_\theta, \widetilde{v}_z \in \widetilde{H}_0^1(\widetilde{\Omega})$. Hence, by (2.8), $v_r|_\Gamma = v_\theta|_\Gamma = v_z|_\Gamma = 0$. Thus, in view of (2.9), to prove (2.7), it suffices to show $v_r|_{\Gamma_0} = v_\theta|_{\Gamma_0} = 0$. Since $v_r, v_\theta \in H_-^1(\Omega)$, by the definition of the norm, we have

$$\int_\Omega r^{-1} v_r^2 \, dr\, dz < \infty \quad \text{and} \quad \int_\Omega r^{-1} v_\theta^2 \, dr\, dz < \infty.$$

By Proposition 3.18 in [3], we have $v_r = v_\theta = 0$ on $\Gamma_0$, which completes the proof. □

**Remark 2.3.** For a bounded domain $\omega \subset \mathbb{R}^3$, define $L_0^2(\omega) := L^2(\omega) \cap \left\{v, \int_\omega v\,dx\,dy\,dz = 0\right\}$. Then, based on the well-posedness of the 3D Stokes problem in $[H_0^1(\widetilde{\Omega})]^3 \times L_0^2(\widetilde{\Omega})$, we consider the solution $(u_r, u_z, p)$ of the ASP in the space $H_{-,0}^1(\Omega) \times H_{+,0}^1(\Omega) \times L_{1,0}^2(\Omega)$, since by Proposition 2.2, these are the proper traces of the solution of the original 3D problem. In addition, for a strong solution $u_z \in H_+^2(\Omega)$, by the definition of the space, we have $r^{-1}\partial_r u_z \in L_1^2(\Omega)$. This implies the Neumann boundary condition $\partial_r u_z = 0$ on $\Gamma_0$. These additional boundary conditions on $\Gamma_0$ ($u_r|_{\Gamma_0} = 0$ from (2.7) and $\partial_r u_z|_{\Gamma_0} = 0$) are due to the axisymmetry of the corresponding 3D vector field.

Let $\boldsymbol{V} := H_{-,0}^1(\Omega) \times H_{+,0}^1(\Omega)$, and $\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 := \|v_r\|_{H_-^1(\Omega)}^2 + \|v_z\|_{H_+^1(\Omega)}^2$ for $\boldsymbol{v} = (v_r, v_z)$. By the trace operators defined in Proposition 2.2, a direct calculation leads to the variational formulation for the ASP (1.1) (also see [6]): Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times L_{1,0}^2(\Omega)$, such that for any $(\boldsymbol{v}, q) \in \boldsymbol{V} \times L_{1,0}^2(\Omega)$,

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, r\,dr\,dz & \text{in} \quad \Omega \\ b(\boldsymbol{u},q) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.10}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\nabla_c \boldsymbol{u} : \nabla_c \boldsymbol{v} + r^{-2} u_r v_r)\, r\,dr\,dz, \quad b(\boldsymbol{u},q) = -\int_\Omega (q\,\mathrm{div}_c \boldsymbol{u} + r^{-1} q u_r)\, r\,dr\,dz, \tag{2.11}$$

$\boldsymbol{u} = (u_r, u_z)^t$, $\boldsymbol{f} = (f_r, f_z)^t$, $\mathrm{div}_c \boldsymbol{u} = \partial_r u_r + \partial_z u_z$, and $\nabla_c \boldsymbol{u} := \begin{pmatrix} \partial_r u_r & \partial_r u_z \\ \partial_z u_r & \partial_z u_z \end{pmatrix}$.

The weak formulation (2.10) is well defined due to the following result.

**Proposition 2.4.** *Let* $H_{+,0}^1(\Omega)'$ *and* $H_{-,0}^1(\Omega)'$ *be the dual spaces of* $H_{+,0}^1(\Omega)$ *and* $H_{-,0}^1(\Omega)$*, respectively, with the pivot space* $L_1^2(\Omega)$*. For* $\boldsymbol{f} \in H_{-,0}^1(\Omega)' \times H_{+,0}^1(\Omega)'$*, the variational form (2.10) defines a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times L_{1,0}^2(\Omega)$ *of the ASP (1.1), and we have*

$$\|u_r\|_{H_-^1(\Omega)} + \|u_z\|_{H_+^1(\Omega)} + \|p\|_{L_1^2(\Omega)} \le C\|\boldsymbol{f}\|_{H_{-,0}^1(\Omega)' \times H_{+,0}^1(\Omega)'}. \tag{2.12}$$

**Proof.** The well-posedness of equation (2.10) can be found in [5,6,19]. The estimate in (2.12) follows directly from the well-posedness. □

We also need the following estimate in subsequent analysis.

**Proposition 2.5.** *For any $v \in H^1_{+,0}(\Omega)$, we have the weighted Poincaré inequality,*

$$\|v\|_{H^1_+(\Omega)} \le C|v|_{H^1_1(\Omega)}.$$

**Proof.** For $v \in H^1_{+,0}(\Omega)$, define $\widetilde{v}(r,\theta,z) = v(r,z)$. Then, by (2.8), $\widetilde{v} \in \widetilde{H}^1_0(\widetilde{\Omega}) \subset H^1_0(\widetilde{\Omega})$. Note that

$$\partial_x = (\cos\theta)\,\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_y = (\sin\theta)\,\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

Thus, by (2.8) and by the Poincaré inequality on the 3D domain $\widetilde{\Omega}$, we have

$$\begin{aligned}\|v\|^2_{H^1_+(\Omega)} &\le C\|\widetilde{v}\|^2_{H^1(\widetilde{\Omega})} \le C|\widetilde{v}|^2_{H^1(\widetilde{\Omega})} = C\int_{\widetilde{\Omega}} |\partial_x\widetilde{v}|^2 + |\partial_y\widetilde{v}|^2 + |\partial_z\widetilde{v}|^2 dxdydz \\ &= C\int_{\widetilde{\Omega}} |\partial_r\widetilde{v}|^2 + |\partial_z\widetilde{v}|^2 rdrd\theta dz = 2\pi C\int_{\Omega} |\partial_r v|^2 + |\partial_z v|^2 rdrdz = 2\pi C|v|^2_{H^1_1(\Omega)},\end{aligned}$$

which completes the proof. □

*2.2. Finite element approximations*

We use mixed finite element methods approximating the ASP (1.1). We first describe the finite element spaces.

Let $\mathcal{T}_h = \{K_i\}$ be a triangulation (partitioning) of the meridian domain $\Omega$, consisting of either triangles or convex quadrilaterals, both shape regular in the sense of [9,11]. Denote by $h < 1$ the maximum diameter of the elements in $\mathcal{T}_h$. For $m \ge 0$ and $K \in \mathcal{T}_h$, let $P_m(K)$ be the space of polynomials of degree $\le m$ on $K$. Let $\hat{K}$ be the reference element of $K$. Namely, when $K$ is a triangle, $\hat{K}$ is a triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$; when $K$ is a convex quadrilateral, $\hat{K} = \{(\hat{r},\hat{z}),\ 0 \le \hat{r},\hat{z} \le 1\}$ is the unit square. On the unit square $\hat{K}$, we define the (tensor-product) polynomial space

$$\hat{Q}_m(\hat{K}) := \text{span}\{p_s(\hat{r})q_t(\hat{z}),\ (\hat{r},\hat{z}) \in \hat{K}\},$$

where $p_s$ and $q_t$ are polynomials of degree $\le m$. If $K \in \mathcal{T}_h$ is a convex quadrilateral, there is a unique bilinear mapping $F_K$, such that $K = F_K(\hat{K})$. Thus, for a quadrilateral $K \in \mathcal{T}_h$, in addition to $P_m(K)$, we denote by $Q_m(K)$ the following space

$$Q_m(K) = \{v(r,z),\ v \circ F_K \in \hat{Q}_m(\hat{K}),\ (r,z) \in K\}.$$

For a triangle $K$, let $K_{\frac{1}{2}} \subset K$ be any of the four sub-triangles obtained by connecting the midpoint on each edge of $K$. Then, to simplify the exposition, we adopt the following notation

$$V_m(K) := \begin{cases} \{v \in C(K),\ v|_{K_{\frac{1}{2}}} \in P_1(K_{\frac{1}{2}})\}, \text{ if } K \text{ is a triangle and } m = 1 \\ P_m(K), \text{ if } K \text{ is a triangle and } m \ge 2 \\ Q_m(K), \text{ if } K \text{ is a quadrilateral and } m \ge 2. \end{cases} \tag{2.13}$$

For $l \ge 0$,

$$R_l(K) := \begin{cases} P_l(K), \text{ if } K \text{ is a triangle} \\ Q_l(K), \text{ if } K \text{ is a quadrilateral.} \end{cases} \tag{2.14}$$

Recall the space $\boldsymbol{V} = H^1_{-,0}(\Omega) \times H^1_{+,0}(\Omega)$. Then, we define the discrete space for the velocity in the ASP.

$$\boldsymbol{V}_h := V^-_h \times V^+_h = \{\boldsymbol{v} = (v_r, v_z) \in \boldsymbol{V},\ v_r|_K, v_z|_K \in V_m(K),\ \forall K \in \mathcal{T}_h\}. \tag{2.15}$$

Note that $\boldsymbol{V} \subset [H^1]^2$ on any region that is away from the $z$-axis. Therefore, $\boldsymbol{V}_h$ in fact consists of continuous vectors, $\boldsymbol{V}_h \subset [C(\Omega)]^2$. For the pressure, we consider the following discrete spaces that are not necessarily continuous: for $l \ge 0$

$$\begin{aligned} P_h &= \{p \in C(\Omega) \cup L^2_{1,0}(\Omega),\ p|_K \in R_l(K),\ \forall K \in \mathcal{T}_h\}; \\ P_h &= \{p \in L^2_{1,0}(\Omega),\ p|_K \in R_l(K),\ \forall K \in \mathcal{T}_h\}; \\ P_h &= \{p \in L^2_{1,0}(\Omega),\ p|_K \in P_l(K),\ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

**Remark 2.6.** The setting above results in different approximation spaces $\boldsymbol{V}_h \times P_h$, depending on the specific selections of $V_m(K)$ and $R_l(K)$. To name a few, this may include the $P_m - P_{m-1}$ $(m \ge 2)$ Hood-Taylor element, the $Q_m - P_{m-1}$ element, and the $P_1\text{iso}P_2 - P_1$ element.

Thus, the mixed finite element method solves the ASP by finding $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$, such that for any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times P_h$,

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h rdrdz \\ b(\boldsymbol{u}_h, q_h) = 0, \end{cases} \tag{2.16}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the bilinear forms defined in (2.11). By the definitions of the spaces $H^1_+(\Omega)$ and $H^1_-(\Omega)$, and by the weighted Poincaré inequality (Proposition 2.5), it is clear that $a(\cdot,\cdot)$ is continuous and coercive on $\boldsymbol{V}_h$. Consequently, for the well-posedness of the discrete problem (2.16), we need to show the inf-sup condition [4,10,15]: There exists $\gamma > 0$, independent of $h$, such that

$$\inf_{0\neq q_h\in P_h}\sup_{0\neq \boldsymbol{v}_h\in \boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_{L^2_1(\Omega)}}\geq \gamma. \tag{2.17}$$

Using macroelement analysis, we shall derive sufficient conditions on the discrete space $\boldsymbol{V}_h\times P_h$, such that this inf-sup condition holds.

We end this section with an important property regarding the divergence operator on axisymmetric fields.

**Lemma 2.7.** *For any $\widetilde{q}\in \widetilde{L}^2(\widetilde{\Omega})$ and $\int_{\widetilde{\Omega}}\widetilde{q}dxdydz=0$, there exists $\widetilde{\boldsymbol{w}}\in \widetilde{\mathbf{H}}^1_0(\widetilde{\Omega})$, such that*

$$\mathrm{div}\widetilde{\boldsymbol{w}}=\widetilde{q},\qquad \text{and}\qquad \|\widetilde{\boldsymbol{w}}\|_{[H^1(\widetilde{\Omega})]^3}\leq C\|\widetilde{q}\|_{L^2(\widetilde{\Omega})}. \tag{2.18}$$

**Proof.** For any $\widetilde{q}\in \widetilde{L}^2(\widetilde{\Omega})$ and $\int_{\widetilde{\Omega}}\widetilde{q}dxdydz=0$, it is clear that $q\in L^2_0(\widetilde{\Omega})$. It is known [2] that there exists $\boldsymbol{w}\in [H^1_0(\widetilde{\Omega})]^3$, which is not necessarily axisymmetric, such that

$$\mathrm{div}\boldsymbol{w}=\widetilde{q},\qquad \text{and}\qquad \|\boldsymbol{w}\|_{[H^1(\widetilde{\Omega})]^3}\leq C\|\widetilde{q}\|_{L^2(\widetilde{\Omega})}. \tag{2.19}$$

Recall the rotation matrix $\mathcal{R}_\sigma$ in (1.3). Then, for $k\in\mathbb{Z}$ and $\sigma,\eta\in\mathbb{R}$, a direct calculation gives

$$\mathcal{R}_{\sigma+2k\pi}=\mathcal{R}_\sigma,\qquad \text{and}\qquad \mathcal{R}_\sigma\mathcal{R}_\eta=\mathcal{R}_\sigma\circ\mathcal{R}_\eta=\mathcal{R}_{\sigma+\eta}. \tag{2.20}$$

Now, define

$$\widetilde{\boldsymbol{w}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{R}_{-\sigma}\boldsymbol{w}\circ\mathcal{R}_\sigma d\sigma. \tag{2.21}$$

Then, for any $\eta\in\mathbb{R}$, by (2.20) and (2.21), we have

$$\begin{aligned}\mathcal{R}_{-\eta}\widetilde{\boldsymbol{w}}\circ\mathcal{R}_\eta&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{R}_{-\eta}\mathcal{R}_{-\sigma}\boldsymbol{w}\circ\mathcal{R}_\sigma\circ\mathcal{R}_\eta d\sigma=\frac{1}{2\pi}\int_{-\pi+\eta}^{\pi+\eta}\mathcal{R}_{-\mu}\boldsymbol{w}\circ\mathcal{R}_\mu d\mu\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{R}_{-\mu}\boldsymbol{w}\circ\mathcal{R}_\mu d\mu=\widetilde{\boldsymbol{w}}.\end{aligned}$$

Thus, by (1.2), $\widetilde{\boldsymbol{w}}$ is axisymmetric. Furthermore, by (2.21) and (2.19), a direct calculation leads to

$$\mathrm{div}\widetilde{\boldsymbol{w}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{div}(\mathcal{R}_{-\sigma}\boldsymbol{w}\circ\mathcal{R}_\sigma)d\sigma=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathrm{div}\boldsymbol{w})\circ\mathcal{R}_\sigma d\sigma=\frac{1}{2\pi}\int_{-\pi}^{\pi}\widetilde{q}\circ\mathcal{R}_\sigma d\sigma=\widetilde{q}, \tag{2.22}$$

where we used the fact that $\widetilde{q}$ is invariant by rotation. By (2.21) and (2.19), it is straightforward to verify that

$$\|\widetilde{\boldsymbol{w}}\|_{[H^1(\widetilde{\Omega})]^3}\leq\|\boldsymbol{w}\|_{[H^1(\widetilde{\Omega})]^3}\leq C\|\widetilde{q}\|_{L^2(\widetilde{\Omega})}.$$

This, together with (2.22), shows the desired result (2.18). □

## 3. Finite element estimates in weighted spaces

Recall that the ASP (1.1) and its finite element approximation (2.16) are defined in weighted spaces (Definition 2.1) that are not equivalent to the usual Sobolev space. In this section, we develop finite element estimates in weighted spaces to study the stability of an interpolation operator to the discrete space. These estimates will be used to derive our main result in Section 4.

Throughout this section, for a sub-region $G\subset\Omega$, we denote

$$r_G:=\max_{(r,z)\in\bar{G}}r\qquad\text{and}\qquad h_G:=\text{the diameter of } G. \tag{3.1}$$

In particular, if $G$ is an edge, $h_G$ denotes its length. Note that in the case that $G$ is the union of several adjacent elements, by the shape regularity of the mesh, we have

$$r_G\simeq\min_{(r,z)\in\bar{G}}r,\quad\text{if } \bar{G}\cap\{r=0\}=\emptyset\qquad\text{and}\qquad r_G\simeq h_G\quad\text{if } \bar{G}\cap\{r=0\}\neq\emptyset.$$

### *3.1. An interpolation operator*

We first define the interpolation operator onto the discrete space $\boldsymbol{V}_h$.

**Definition 3.1.** Denote by $x_i=(r_i,z_i)$ the $i$th Lagrange node associated with the space $\boldsymbol{V}_h$. Note that in the case $m=1$ with the triangular mesh, the midpoint of each edge in the triangulation $\mathcal{T}_h$ is also a Lagrange node. Let $N=\{x_i\}$ be the union of these nodal points. Then, for any $\boldsymbol{v}=(v_r,v_z)\in\boldsymbol{V}$, we define $\mathcal{I}:\boldsymbol{V}\to\boldsymbol{V}_h$, such that $\mathcal{I}\boldsymbol{v}=(\mathcal{I}^-v_r,\mathcal{I}^+v_z)\in\boldsymbol{V}_h$, where $\mathcal{I}^-$ and $\mathcal{I}^+$ are defined as follows to take into account the boundary conditions in $H^1_{-,0}(\Omega)$ and $H^1_{+,0}(\Omega)$.

(I) (Interpolation at Boundary Nodes). For a node $x_i\in N$, we say it is a boundary node for $\mathcal{I}^-$ if $x_i\in\partial\Omega$, and say it is a boundary node for $\mathcal{I}^+$ if $x_i\in\Gamma$. Otherwise, we call it an interior node. Then we define the interpolation at the boundary nodes:

$$\mathcal{I}^+ v(x_i) = 0, \quad \text{for } x_i \in \Gamma,\ \forall v \in H^1_{+,0}(\Omega);$$
$$\mathcal{I}^- v(x_i) = 0, \quad \text{for } x_i \in \partial\Omega,\ \forall v \in H^1_{-,0}(\Omega).$$

We now proceed to define the interpolation at interior nodes.

(II) (Interpolation at Interior Nodes). Let $x_i \in N$ be an interior node defined above. Denote by $S_i$ the support of the basis function associated with $x_i$. Define

$$|S_i| := \int_{S_i} r dr dz. \tag{3.2}$$

For each edge $e_j$ of the triangulation $\mathcal{T}_h$, we say $e_j$ is an interior edge if $e_j \notin \partial\Omega$. Recall the space $V_m$ in (2.13). Then, for each interior edge $e_j$, we select a node $x_j^e \in N$ in the interior of $e_j$, such that

$$\int_{e_j} \phi_j^e r ds \simeq r_{e_j} h_{e_j}, \tag{3.3}$$

where $\phi_j^e$ is the Lagrange basis function associated with $x_j^e$. This is possible as shown below. For $m = 1$ and a triangular mesh, (3.3) clearly holds by the scaling argument and shape regularity of the mesh. For $m \geq 2$, choose the nodes on $e_j$ to be located on the endpoints and on the (modified) Gaussian quadrature points. Note that for both triangular and quadrilateral elements, $\phi_j^e r|_{e_j}$ is a polynomial of degree $\leq m+1$ ($m \geq 2$). Choose $x_j^e$ to be a quadrature point in the interior of $e_j$. Then, by the scaling argument, (3.3) holds for both elements. Let $N^e := \{x_j^e\}$ be the collection of such nodes and let $N'$ be the collection of all the other interior nodes. Then, we define the interpolation at the interior nodes: for $x_i \in N'$,

$$\mathcal{I}^+ v(x_i) = \frac{\int_{S_i} v r dr dz}{|S_i|}, \quad \forall v \in H^1_{+,0}(\Omega) \quad \text{and} \quad \mathcal{I}^- v(x_i) = \frac{\int_{S_i} v r dr dz}{|S_i|}, \quad \forall v \in H^1_{-,0}(\Omega); \tag{3.4}$$

for $x_j^e \in N^e$, we choose the value of $\mathcal{I}\boldsymbol{v}(x_j^e)$, such that

$$\int_{e_j} (\mathcal{I}^+ v) r ds = \int_{e_j} v r ds, \quad \forall v \in H^1_{+,0}(\Omega) \quad \text{and} \quad \int_{e_j} (\mathcal{I}^- v) r ds = \int_{e_j} v r ds, \quad \forall v \in H^1_{-,0}(\Omega). \tag{3.5}$$

The definition (3.5) makes sense, since the values of $\mathcal{I}^+ v$ and $\mathcal{I}^- v$ at other nodes on $e_j$ have been determined by either (3.4) or in (I) above as a boundary node.

**Remark 3.2.** Based on Definition 3.1, the sets of interior nodes and boundary nodes are different for the interpolation operators $\mathcal{I}^-$ and $\mathcal{I}^+$. Therefore, although the equations in (3.4) and (3.5) are the same for both interpolation operators, the corresponding interior node sets $N'$ and $N^e$ may be different.

For $K \in \mathcal{T}_h$, let $\boldsymbol{n}_K$ be the outward-pointing unit normal vector of $\partial K$. Let $q$ be a piecewise constant function such that $q|_K \in \mathbb{R}$. Then, by (3.5) and integration by parts, the interpolation operator satisfies

$$\begin{aligned} b(\mathcal{I}\boldsymbol{v}, q) &= -\sum_{K\in\mathcal{T}_h} \int_K q \operatorname{div}_c(r\mathcal{I}\boldsymbol{v}) dr dz = -\sum_{K\in\mathcal{T}_h} \int_{\partial K} q r \mathcal{I}\boldsymbol{v} \cdot \boldsymbol{n}_K ds \\ &= -\sum_{K\in\mathcal{T}_h} \int_{\partial K} q r \boldsymbol{v} \cdot \boldsymbol{n}_K ds = b(\boldsymbol{v}, q). \end{aligned} \tag{3.6}$$

In the rest of this section, we shall show that the equations in (3.4) and (3.5) are well defined for $\boldsymbol{v} \in \boldsymbol{V} = H^1_{-,0}(\Omega) \times H^1_{+,0}(\Omega)$, and therefore, the interpolation $\mathcal{I}\boldsymbol{v} = (\mathcal{I}^- v_r, \mathcal{I}^+ v_z)$ is uniquely determined.

### *3.2. Some lemmas*

We start by recalling a trace estimate (Lemma A1 in [12]). We say a triangle $T$ is of type I if it has only one vertex on the $z$-axis; and $T$ is of type II if it has two vertices on the $z$-axis. Then, we have

**Lemma 3.3.** *If $e$ is the edge of a type I triangle $T$, then,*

$$\|vr\|^2_{L^2(e)} \leq C(\|v\|^2_{L^2_1(T)} + h_e^2 |v|^2_{H^1_1(T)});$$

*if $e$ is the edge of a type II triangle $T$,*

$$\|v\|^2_{L^2_1(e)} \leq C(h_e^{-1} \|v\|^2_{L^2_1(T)} + h_e |v|^2_{H^1_1(T)}),$$

*where $v \in H^1_1(T)$.*

Meanwhile, we shall also need the following estimate in the $H^1_-$ norm.

**Lemma 3.4.** *For any $v \in H^1_0(\Omega)$, we have $v \in H^1_{-,0}(\Omega)$, and $\|r^{-1} \cdot\|_{L^2_1(\Omega)}$ defines a norm for $v$.*

**Proof.** Following a direct calculation, we have

$$|v|^2_{H^1_1(\Omega)} = \int_\Omega (|\partial_r v|^2 + |\partial_z v|^2) r dr dz \le C \int_\Omega |\partial_r v|^2 + |\partial_z v|^2 dr dz = C|v|^2_{H^1(\Omega)}.$$

Then, it remains to show $\|r^{-2}v\|_{L^2_1(\Omega)}$ is bounded. Note that the following regularity estimate ([23,24]) holds for $v \in H^1_0(\Omega)$,

$$\int_\Omega r^{-2} v^2 dr dz \le C|v|^2_{H^1(\Omega)}. \tag{3.7}$$

Using (3.7), we have

$$\|r^{-1}v\|^2_{L^2_1(\Omega)} = \int_\Omega r^{-1} v^2 dr dz \le \int_\Omega r^{-2} v^2 dr dz \le C|v|^2_{H^1(\Omega)}.$$

This completes the proof. □

Based on scaling arguments in weighted spaces, we now give estimates on the Lagrange basis functions in the discrete velocity space (2.15).

**Lemma 3.5.** *For $K \in \mathcal{T}_h$, recall $r_K$ and $h_K$ from (3.1). Let $\phi$ be a Lagrange basis function, such that $K \subset \operatorname{supp}\phi$. Then, for $\ell = 0, 1$, we have*

$$|\phi|_{H^\ell_1(K)} \simeq r_K^{1/2} h_K^{1-\ell}. \tag{3.8}$$

*In addition, if $\phi$ vanishes on the $z$-axis, then*

$$\|r^{-1}\phi\|_{L^2_1(K)} \simeq r_K^{-1/2} h_K. \tag{3.9}$$

**Proof.** In the case when $K$ is a triangle, see Lemma A.3 and Lemma A.4 in [19] for the proofs of these estimates. Thus, we concentrate on the proof when $K$ is a quadrilateral.

If $K \cap \{r = 0\} = \emptyset$, (3.8) and (3.9) can be derived from the scaling argument for quadrilateral elements in $H^\ell$ and the definition of the spaces involved,

$$|\phi|_{H^\ell_1(K)} \simeq r_K^{1/2} |\phi|_{H^\ell(K)} \simeq r_K^{1/2} h_K^{1-\ell}, \qquad \|r^{-1}\phi\|_{L^2_1(K)} \simeq r_K^{-1/2} \|\phi\|_{L^2(K)} \simeq r_K^{-1/2} h_K.$$

If $K \cap \{r = 0\} \ne \emptyset$, let $v_i$, $1 \le i \le 4$ be the vertices of $K$. Let the bilinear mapping $F_K : \hat{K} \to K$ be such that it maps vertices to vertices as follows:

$$(0,0) \to v_1,\ (1,0) \to v_2,\ (1,1) \to v_3,\ (0,1) \to v_4. \tag{3.10}$$

Note that its Jacobian $|J_{F_K}| \simeq h_K^2$ [15]. Let $r_i$, $1 \le i \le 4$ be the distances from $v_i$ to the $z$-axis. Then, we consider two possible cases:

[I] ($K \cap \{r = 0\}$ = a vertex). Without loss of generality, assume $v_1$ is on the $z$-axis, and therefore $r_1 = 0$. Thus, the bilinear mapping $F_K$ for $r$ reads

$$r = r_2\hat{r} + r_4\hat{z} + (r_3 - r_4 - r_2)\hat{r}\hat{z}, \qquad \text{for } (r,z) \in K,\ (\hat{r},\hat{z}) \in \hat{K}.$$

Based on the shape regularity of the quadrilateral, we have $r_3 > \min(r_2, r_4)$ and $r_2 \simeq r_3 \simeq r_4 \simeq h_K$. Therefore, since $0 \le \hat{r}, \hat{z} \le 1$, we have

$$c_1 h_K(\hat{r} + \hat{z}) \le r_2\hat{r} + r_4\hat{z} + (r_3 - r_4 - r_2)\hat{r}\hat{z} \le c_2 h_K(\hat{r} + \hat{z}), \tag{3.11}$$

where $c_1, c_2 > 0$ depend on the shape regularity of the quadrilateral, but not on $h_K$. Let $\hat{\phi} := \phi \circ F_K$. Then, by the estimate on the Jacobian $|J_{F_K}|$ and (3.11), we have

$$\begin{aligned}\int_K \phi^2 r^{-1} dr dz &\simeq h_K^2 \int_{\hat{K}} \hat{\phi}^2 [r_2\hat{r} + r_4\hat{z} + (r_3 - r_4 - r_2)\hat{r}\hat{z}]^{-1} d\hat{r} d\hat{z} \\ &\simeq h_K \int_{\hat{K}} \hat{\phi}^2 (\hat{r} + \hat{z})^{-1} d\hat{r} d\hat{z}, \end{aligned} \tag{3.12}$$

and for $\ell = 0, 1$,

$$\begin{aligned}\int_K |\nabla^\ell_{r,z}\phi|^2 r dr dz &\simeq h_K^2 h_K^{-2\ell} \int_{\hat{K}} |\nabla^\ell_{\hat{r},\hat{z}}\hat{\phi}|^2 (r_2\hat{r} + r_4\hat{z} + (r_3 - r_4 - r_2)\hat{r}\hat{z}) d\hat{r} d\hat{z} \\ &\simeq h_K^{3-2\ell} \int_{\hat{K}} |\nabla^\ell_{\hat{r},\hat{z}}\hat{\phi}|^2 (\hat{r} + \hat{z}) d\hat{r} d\hat{z}. \end{aligned} \tag{3.13}$$

Note that $\phi \in H^1_0(\Omega)$. Then, by Lemma 3.4 and (3.12),

$$\int_{\hat{K}} \hat{\phi}^2 (\hat{r} + \hat{z})^{-1} d\hat{r} d\hat{z} \simeq h_K^{-1} \int_K \phi^2 r^{-1} dr dz < \infty.$$

Therefore, $\left(\int_{\hat{K}} \hat{\phi}^2 (\hat{r} + \hat{z})^{-1} d\hat{r} d\hat{z}\right)^{1/2}$ defines a norm of $\hat{\phi}$ on $\hat{K}$. Thus, by (3.12) and the norm equivalence in finite-dimensional spaces, we have

$$\|r^{-1}\phi\|^2_{L^2_1(K)} \simeq h_K \int_{\hat{K}} \hat{\phi}^2(\hat{r}+\hat{z})^{-1} d\hat{r}d\hat{z} \simeq h_K \int_{\hat{K}} \hat{\phi}^2 d\hat{r}d\hat{z} \simeq h_K \simeq r_K^{-1} h_K^2,$$

where we used the fact $r_K \simeq h_K$. This proves the estimate (3.9) in this case. For the estimate (3.8), with a similar process using (3.13) and the norm equivalence in the finite-dimensional spaces, we have

$$|\phi|^2_{H^\ell_1(K)} \simeq h_K^{3-2\ell} \int_{\hat{K}} |\nabla^\ell_{\hat{r},\hat{z}}\hat{\phi}|^2(\hat{r}+\hat{z}) d\hat{r}d\hat{z} \simeq h_K^{3-2\ell} \int_{\hat{K}} |\nabla^\ell_{\hat{r},\hat{z}}\hat{\phi}|^2 d\hat{r}d\hat{z} \simeq h_K^{3-2\ell} \simeq r_K h_K^{2-2\ell}. \tag{3.14}$$

[II] ($K \cap \{r=0\}$ = an edge). The proof in this case follows similarly as in case [I] above. Assume the vertices $v_1$ and $v_2$ are on the $z$-axis, and therefore the distances $r_1 = r_2 = 0$. Thus, the bilinear mapping $F_K$ (3.10) for $r$ reads

$$r = r_4\hat{z} + (r_3 - r_4)\hat{r}\hat{z}, \qquad \text{for } (r,z)\in K,\ (\hat{r},\hat{z})\in\hat{K}.$$

Then, by the shape regularity of the quadrilateral, we have $r_3 \simeq r_4 \simeq h_K$ and

$$c_1 h_K \hat{z} \le r_4\hat{z} + (r_3 - r_4)\hat{r}\hat{z} \le c_2 h_K \hat{z}, \tag{3.15}$$

where $c_1, c_2 > 0$ depend on the shape regularity of the quadrilateral, but not on $h_K$. Then, by the estimate on the Jacobian $|J_{F_K}|$ and (3.15), we have

$$\int_K \phi^2 r^{-1} dr dz \simeq h_K^2 \int_{\hat{K}} \hat{\phi}^2 (r_4\hat{z} + (r_3-r_4)\hat{r}\hat{z})^{-1} d\hat{r}d\hat{z} \simeq h_K \int_{\hat{K}} \hat{\phi}^2 \hat{z}^{-1} d\hat{r}d\hat{z}. \tag{3.16}$$

Since $\phi \in H^1_0(\Omega)$, by Lemma 3.4 and (3.16), we have $\int_{\hat{K}} \hat{\phi}^2\hat{z}^{-1} d\hat{r}d\hat{z} < \infty$, and therefore $\left(\int_{\hat{K}} \hat{\phi}^2\hat{z}^{-1} d\hat{r}d\hat{z}\right)^{1/2}$ defines a norm of $\hat{\phi}$ on $\hat{K}$. Thus, by (3.16) and the norm equivalence in finite-dimensional spaces, we have

$$\|r^{-1}\phi\|^2_{L^2_1(K)} \simeq h_K \int_{\hat{K}} \hat{\phi}^2 \hat{z}^{-1} d\hat{r}d\hat{z} \simeq h_K \int_{\hat{K}} \hat{\phi}^2 d\hat{r}d\hat{z} \simeq h_K \simeq r_K^{-1} h_K^2.$$

Thus, we have the desired estimate in (3.9) for this case. By the relation in (3.15), the estimate (3.8) can be shown using a similar calculation as in (3.13) and (3.14).

Hence, we have completed the proof. □

**Remark 3.6.** Without additional difficulties, the scaling argument, especially (3.13) and (3.14) in the proof of (3.8), extends to the case when $\phi = 1$ and $\ell = 0$. This leads to

$$\int_K r dr dz \simeq r_K h_K^2, \quad \forall K \in \mathcal{T}_h, \tag{3.17}$$

which we shall also need to carry out further analysis.

### *3.3. Stability of the interpolation*

We proceed to analyze the stability of the interpolation operator (Definition 3.1) in weighted spaces. The main result of this section is summarized in Theorem 3.10.

We first have the pointwise estimate.

**Lemma 3.7.** *Recall the interpolation operator $\mathcal{I}$, the support $S_i$, and the sets ($N, N'$, and $N^e$) from Definition 3.1. For a subdomain $G \subset \Omega$, recall $r_G$ and $h_G$ from (3.1). Then, for $x_i \in N'$, we have*

$$\begin{aligned} |\mathcal{I}^+ v(x_i)| &\le C r_{S_i}^{-1/2} h_{S_i}^{-1} \|v\|_{L^2_1(S_i)}, \quad \forall v \in H^1_{+,0}(\Omega), \\ |\mathcal{I}^- v(x_i)| &\le C r_{S_i}^{-1/2} h_{S_i}^{-1} \|v\|_{L^2_1(S_i)}, \quad \forall v \in H^1_{-,0}(\Omega). \end{aligned} \tag{3.18}$$

*For $x_i^e \in N^e$, suppose $x_i^e$ is a node on $K_i \in \mathcal{T}_h$. Let $U_{K_i}$ be the union of elements adjacent to $K_i$. Then,*

$$\begin{aligned} |\mathcal{I}^+ v(x_i^e)| &\le C\left(r_{U_{K_i}}^{-1/2} h_{U_{K_i}}^{-1} \|v\|_{L^2_1(U_{K_i})} + r_{U_{K_i}}^{-1/2} |\boldsymbol{v}|_{H^1_1(U_{K_i})}\right), \quad \forall v \in H^1_{+,0}(\Omega), \\ |\mathcal{I}^- v(x_i^e)| &\le C\left(r_{U_{K_i}}^{-1/2} h_{U_{K_i}}^{-1} \|v\|_{L^2_1(U_{K_i})} + r_{U_{K_i}}^{-1/2} |\boldsymbol{v}|_{H^1_1(U_{K_i})}\right), \quad \forall v \in H^1_{-,0}(\Omega). \end{aligned} \tag{3.19}$$

**Proof.** We here show the proof for the operator $\mathcal{I}^+$ and $v \in H^1_{+,0}(\Omega)$.

For $x_i \in N'$, by (3.4),

$$\mathcal{I}^+ v(x_i) = \frac{\int_{S_i} v r dr dz}{|S_i|}.$$

In the case $S_i \cap \{r=0\} = \emptyset$, by (3.2), shape regularity of the mesh, and Hölder's inequality, we have

$$|\mathcal{I}^+ v(x_i)| = \left|\frac{\int_{S_i} v r dr dz}{|S_i|}\right| \le C \frac{\int_{S_i} |v| dr dz}{\int_{S_i} dr dz} \le C h_{S_i}^{-1} \|v\|_{L^2(S_i)} \le C r_{S_i}^{-1/2} h_{S_i}^{-1} \|v\|_{L^2_1(S_i)}.$$

In the case $S_i \cap \{r=0\} \neq \emptyset$, by Hölder's inequality and (3.17), we have

$$|\mathcal{I}^+ v(x_i)| = |\frac{\int_{S_i} vrdrdz}{|S_i|}| \le \frac{\|r^{1/2}\|_{L^2(S_i)}\|v\|_{L^2_1(S_i)}}{\int_{S_i} rdrdz} \le C r_{S_i}^{-1/2} h_{S_i}^{-1} \|v\|_{L^2_1(S_i)}.$$

Therefore, we have proved (3.18).

For $x_i^e \in N^e$, recall the Lagrange basis function $\phi_i^e$ from Definition 3.1. Then, by (3.5), we have

$$\int_{e_i} \mathcal{I}^+ v(x_i^e)\phi_i^e r ds = \int_{e_i} v r ds - \sum_{x_k \in e_i \cap N'} \int_{e_i} \mathcal{I}^+ v(x_k)\phi_k r ds. \tag{3.20}$$

In the case $e_i \cap \{r=0\} = \emptyset$, suppose $e_i$ is the edge of $K_i \in \mathcal{T}_h$. Then, by (3.3), (3.20), Hölder's inequality, (3.18), the scaling argument, the fact $h_{e_i} \simeq h_{S_k}$, and the trace theorem, we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| r_{e_i} h_{e_i} &\simeq |\int_{e_i} \mathcal{I}^+ v(x_i)\phi_i^e r ds| \le |\int_{e_i} v r ds| + \sum_{x_k \in e_i \cap N'} |\mathcal{I}^+ v(x_k)\int_{e_i} \phi_k r ds| \\
&\le C r_{e_i}\big(h_{e_i}^{1/2}\|v\|_{L^2(e_i)} + \sum_{x_k \in e_i \cap N'} r_{S_k}^{-1/2} h_{S_k}^{-1}\|v\|_{L^2_1(S_k)} h_{e_i}\big) \\
&\le C r_{e_i}\big(h_{e_i}^{1/2}(h_{e_i}^{-1/2}\|v\|_{L^2(K_i)} + h_{e_i}^{1/2}|v|_{H^1(K_i)}) + \sum_{x_k \in e_i \cap N'} r_{S_k}^{-1/2}\|v\|_{L^2_1(S_k)}\big) \\
&\le C r_{e_i}\big(r_{e_i}^{-1/2} h_{e_i}^{1/2}(h_{e_i}^{-1/2}\|v\|_{L^2_1(K_i)} + h_{e_i}^{1/2}|v|_{H^1_1(K_i)}) + \sum_{x_k \in e_i \cap N'} r_{S_k}^{-1/2}\|v\|_{L^2_1(S_k)}\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| &\le C\big(r_{e_i}^{-1/2} h_{e_i}^{-1}\|v\|_{L^2_1(U_{K_i})} + r_{e_i}^{-1/2}|v|_{H^1_1(U_{K_i})}\big) \\
&\le C\big(r_{U_{K_i}}^{-1/2} h_{U_{K_i}}^{-1}\|v\|_{L^2_1(U_{K_i})} + r_{U_{K_i}}^{-1/2}|v|_{H^1_1(U_{K_i})}\big).
\end{aligned} \tag{3.21}$$

In the case $e_i \cap \{r=0\} \neq \emptyset$, we have $r_{e_i} \simeq h_{e_i}$. Therefore, by (3.3), (3.20), Hölder's inequality, (3.18), and the scaling argument, we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| r_{e_i} h_{e_i} &\simeq |\mathcal{I}^+ v(x_i^e)||\int_{e_i} \phi_i^e r ds| \le |\int_{e_i} v r ds| + \sum_{x_k \in e_i \cap N'} |\mathcal{I}^+ v(x_k)\int_{e_i} \phi_k r ds| \\
&\le C\big(|\int_{e_i} v r ds| + \sum_{x_k \in e_i \cap N'} r_{S_k}^{-1/2} h_{S_k}^{-1}\|v\|_{L^2_1(S_k)} r_{e_i} h_{e_i}\big).
\end{aligned} \tag{3.22}$$

Then, we use the weighed trace estimate in Lemma 3.3 to evaluate the term $|\int_{e_i} v r ds|$ as follows. 1) For a triangular mesh, we let $T = K \in \mathcal{T}_h$ be a triangle with $e_i$ as an edge. 2) For a quadrilateral mesh, let $K \in \mathcal{T}_h$ be a quadrilateral with $e_i$ as an edge. Decompose $K$ using one of its diagonals into two triangles. Suppose $T \subset K$ is one of the two triangles that has $e_i$ as an edge. Note that $r_{e_i} \simeq h_{e_i} \simeq r_{S_i} \simeq h_{S_i}$. In both cases, if $T$ is of type I, by (3.22), Hölder's inequality, and Lemma 3.3, we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| h_{e_i}^2 &\le C\big(h_{e_i}^{1/2}\|vr\|_{L^2(e_i)} + \sum_{x_k \in e_i \cap N'} r_{e_i}^{1/2}\|v\|_{L^2_1(S_k)}\big) \\
&\le C\big(h_{e_i}^{1/2}(\|v\|_{L^2_1(T)} + h_{e_i}|v|_{H^1_1(T)}) + \sum_{x_k \in e_i \cap N'} h_{e_i}^{1/2}\|v\|_{L^2_1(S_k)}\big);
\end{aligned} \tag{3.23}$$

if $T$ is of type II, by (3.22), Hölder's inequality, the scaling argument, and Lemma 3.3, we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| h_{e_i}^2 &\le C\big(\|r^{1/2}\|_{L^2(e_i)}\|v\|_{L^2_1(e_i)} + \sum_{x_k \in e_i \cap N'} r_{e_i}^{1/2}\|v\|_{L^2_1(S_k)}\big) \\
&\le C\big(h_{e_i}(h_{e_i}^{-1/2}\|v\|_{L^2_1(T)} + h_{e_i}^{1/2}|v|_{H^1_1(T)}) + \sum_{x_k \in e_i \cap N'} h_{e_i}^{1/2}\|v\|_{L^2_1(S_k)}\big).
\end{aligned} \tag{3.24}$$

Therefore, by (3.23) and (3.24), we have

$$\begin{aligned}
|\mathcal{I}^+ v(x_i^e)| &\le C\big(h_{e_i}^{-3/2}\|v\|_{L^2_1(T)} + h_{e_i}^{-1/2}|v|_{H^1_1(T)} + \sum_{x_k \in e_i \cap N'} h_{e_i}^{-3/2}\|v\|_{L^2_1(S_k)}\big) \\
&\le C\big(h_{e_i}^{-3/2}\|v\|_{L^2_1(U_{K_i})} + h_{e_i}^{-1/2}|v|_{H^1_1(U_{K_i})}\big) \le C\big(r_{U_{K_i}}^{-1/2} h_{U_{K_i}}^{-1}\|v\|_{L^2_1(U_{K_i})} + r_{U_{K_i}}^{-1/2}|v|_{H^1_1(U_{K_i})}\big).
\end{aligned}$$

This, together with (3.21), completes the proof for (3.19).

We skip the estimate for $\mathcal{I}^-$ and $v \in H^1_{-,0}(\Omega)$, since it follows similarly. □

Then, we obtain the stability estimate on each element.

**Lemma 3.8.** *For a given $K \in \mathcal{T}_h$, let $U_K$ be the union of elements adjacent to $K$. Let $\ell = 0, 1$. Then, the interpolation operators satisfy*

$$|\mathcal{I}^+ v|_{H_1^\ell(K)} \le C(h_K^{1-\ell}|v|_{H_1^1(U_K)} + h_K^{-\ell}\|v\|_{L_1^2(U_K)}), \quad \forall v \in H_{+,0}^1(\Omega); \tag{3.25}$$

$$|\mathcal{I}^- v|_{H_1^\ell(K)} \le C(h_K^{1-\ell}|v|_{H_1^1(U_K)} + h_K^{-\ell}\|v\|_{L_1^2(U_K)}), \quad \forall v \in H_{-,0}^1(\Omega). \tag{3.26}$$

*In addition, for $v \in H_{-,0}^1(\Omega)$, we have*

$$\|r^{-1}\mathcal{I}^- v\|_{L_1^2(K)} \le C\|v\|_{H_-^1(U_K)}. \tag{3.27}$$

**Proof.** Let $x_i$ be a node in $K$. For $x_i \in N'$, note that the support of the associated basis function $S_i = \operatorname{supp}\phi_i \subset U_K$. Then, by the estimates in (3.18) and (3.8), we have

$$|\mathcal{I}^+ v(x_i)\phi_i|_{H_1^\ell(K)} \le |\mathcal{I}^+ v(x_i)||\phi_i|_{H_1^\ell(K)} \le Cr_K^{1/2}h_K^{1-\ell}r_{U_K}^{-1/2}h_{U_K}^{-1}\|v\|_{L_1^2(U_K)} \le Ch_K^{-\ell}\|v\|_{L_1^2(U_K)}. \tag{3.28}$$

For $x_i \in N^e$, by the estimates in (3.19) and (3.8), we have

$$\begin{aligned}|\mathcal{I}^+ v(x_i)\phi_i|_{H_1^\ell(K)} &\le |\mathcal{I}^+ v(x_i)||\phi_i|_{H_1^\ell(K)} \le Cr_K^{1/2}h_K^{1-\ell}\big(r_{U_K}^{-1/2}h_{U_K}^{-1}\|v\|_{L_1^2(U_K)} + r_{U_K}^{-1/2}|v|_{H_1^1(U_K)}\big)\\ &\le C\big(h_K^{-\ell}\|v\|_{L_1^2(U_K)} + h_K^{1-\ell}|v|_{H_1^1(U_K)}\big).\end{aligned} \tag{3.29}$$

Thus, combining (3.28) and (3.29), we have proved the estimate (3.25),

$$|\mathcal{I}^+ v|_{H_1^\ell(K)} \le \sum_{x_i \in K}|\mathcal{I}^+ v(x_i)\phi_i|_{H_1^\ell(K)} \le C\big(h_K^{-\ell}\|v\|_{L_1^2(U_K)} + h_K^{1-\ell}|v|_{H_1^1(U_K)}\big).$$

We skip the proof of (3.26) for $\mathcal{I}^- v$, since it follows from a similar calculation based on the corresponding estimates in Lemma 3.7 and (3.8).

Now, it suffices to show the estimate (3.27) to complete the proof. For $v \in H_{-,0}^1(\Omega)$, if $K \cap \{r = 0\} = \emptyset$, by the definition of the norm, (3.26), and $h_K \le Cr_K$, we have

$$\|r^{-1}\mathcal{I}^- v\|_{L_1^2(K)} \simeq r_K^{-1}\|\mathcal{I}^- v\|_{L_1^2(K)} \le Cr_K^{-1}\big(\|v\|_{L_1^2(U_K)} + h_K|v|_{H_1^1(U_K)}\big) \le C\|v\|_{H_-^1(U_K)}. \tag{3.30}$$

If $K \cap \{r = 0\} \ne \emptyset$, by the estimates in (3.9), the corresponding estimates in Lemma 3.7, and Definition 3.1 we have

$$\begin{aligned}\|r^{-1}\mathcal{I}^- v\|_{L_1^2(K)} &\le \sum_{x_i \in K\setminus\{r=0\}}|\mathcal{I}^{-1}v(x_i)|\|r^{-1}\phi_i\|_{L_1^2(K)} \le Cr_K^{-1/2}h_K\sum_{x_i \in K\setminus\{r=0\}}|\mathcal{I}^- v(x_i)|\\ &\le Cr_K^{-1/2}h_K\big(r_{U_K}^{-1/2}h_{U_k}^{-1}\|v\|_{L_1^2(U_K)} + r_{U_K}^{-1/2}|v|_{H_1^1(U_K)}\big) \le C\|v\|_{H_-^1(U_K)}.\end{aligned}$$

This, together with (3.30), completes the proof of (3.27), and hence the proof of the lemma. □

The results in Lemma 3.8 can be further refined and extended to the following global estimates.

**Lemma 3.9.** *Recall the interpolation operator $\mathcal{I}$ from Definition 3.1. Then, for $v \in H_{+,0}^1(\Omega)$ we have*

$$|\mathcal{I}^+ v|_{H_1^1(\Omega)} \le C|v|_{H_1^1(\Omega)}, \tag{3.31}$$

*and for $v \in H_{-,0}^1(\Omega)$,*

$$|\mathcal{I}^- v|_{H_1^1(\Omega)} \le C\|v\|_{H_-^1(\Omega)}. \tag{3.32}$$

**Proof.** Let $\Pi$ denote either $\mathcal{I}^+$ or $\mathcal{I}^-$, depending on the underlying function. Then, we have

$$|\Pi v|_{H_1^1(\Omega)}^2 \le |v|_{H_1^1(\Omega)}^2 + |v - \Pi v|_{H_1^1(\Omega)}^2 \le |v|_{H_1^1(\Omega)}^2 + \sum_{K \in \mathcal{T}_h}|v - \Pi v|_{H_1^1(K)}^2. \tag{3.33}$$

For any $K \in \mathcal{T}_h$, let $U_K$ be the union of elements adjacent to $K$, and let $\mathbb{U}_K$ be the union of elements adjacent to $U_K$. Recall the discrete velocity space $\boldsymbol{V}_h = V_h^- \times V_h^+$. We let $V$ denote either $V_h^+$ or $V_h^-$ in the following estimates since the analysis is suitable for either space. Let $p \in V$ be a function, such that $p =$ constant on $U_K$. Then, based on the definition of $\Pi$, it is straightforward to verify $\Pi p|_K = p|_K$. Thus, by (3.25) and (3.26), we have

$$\begin{aligned}|v - \Pi v|_{H_1^1(K)} &\le |v - p|_{H_1^1(K)} + |\Pi(v - p)|_{H_1^1(K)} \le C(|v - p|_{H_1^1(U_K)} + h_K^{-1}\|v - p\|_{L_1^2(U_K)})\\ &\le C(|v|_{H_1^1(U_K)} + h_K^{-1}\|v - p\|_{L_1^2(U_K)}).\end{aligned} \tag{3.34}$$

Therefore, by (3.33) and (3.34), in order to prove (3.31) and (3.32), it suffices to show $h_K^{-1}\|v - p\|_{L_1^2(U_K)} \le C|v|_{H_1^1(\mathbb{U}_K)}$ for $v \in H_{+,0}^1(\Omega)$, and $h_K^{-1}\|v - p\|_{L_1^2(U_K)} \le C\|v\|_{H_-^1(\mathbb{U}_K)}$ for $v \in H_{-,0}^1(\Omega)$. The proof is thus separated to these two cases.

Case I ($v \in H_{+,0}^1(\Omega)$). We need to consider three situations.

[I] $U_K \cap \partial\Omega = \emptyset$. We choose $p|_{U_K} =$ the $L^2$ projection of $v$ on the constant function space over $U_K$. Then, using the polynomial approximation property in Sobolev spaces and $h_K \simeq h_{U_K}$, we have

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} \le Ch_K^{-1} r_{U_K}^{1/2}\|v-p\|_{L^2(U_K)} \le Ch_K^{-1} r_{U_K}^{1/2} h_{U_K}|v|_{H^1(U_K)} \le C|v|_{H_1^1(U_K)}. \tag{3.35}$$

[II] $U_K \cap \Gamma \neq \emptyset$. We choose $p|_{U_K}=0$. Note that $\mathbb{U}_K \cap \Gamma$ always has positive measure. Thus, if $\mathbb{U}_K \cap \{r=0\}=\emptyset$, using the Poincaré inequality, the scaling argument, and $h_K \simeq h_{\mathbb{U}_K}$, we have

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} = h_K^{-1}\|v\|_{L_1^2(U_K)} \le Ch_K^{-1} r_{U_K}^{1/2}\|v\|_{L^2(U_K)} \le Cr_{U_K}^{1/2}|v|_{H^1(\mathbb{U}_K)} \le C|v|_{H_1^1(\mathbb{U}_K)}. \tag{3.36}$$

If $\mathbb{U}_K \cap \{r=0\} \neq \emptyset$, let $\widetilde{\mathbb{U}}_K$ be the 3D region obtained by the rotation of $\mathbb{U}_K$ about the $z$-axis. Define $\widetilde{v}(r,\theta,z)=v(r,z)$ for $(r,\theta,z)\in\widetilde{\mathbb{U}}_K$. Then, by Proposition 2.2, the Poincaré inequality on $\widetilde{\mathbb{U}}_K$, and the scaling argument, we have

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} = h_K^{-1}\|v\|_{L_1^2(U_K)} \le Ch_K^{-1}\|\widetilde{v}\|_{L^2(\widetilde{\mathbb{U}}_K)} \le C|\widetilde{v}|_{H^1(\widetilde{\mathbb{U}}_K)} \le C|v|_{H_1^1(\mathbb{U}_K)}. \tag{3.37}$$

[III] $U_K \cap \Gamma = \emptyset$ and $U_K \cap \{r=0\} \neq \emptyset$. We choose $p \in V_h^+$, such that $\int_{U_K} prdrdz = \int_{U_K} vrdrdz$. Let $\widetilde{U}_K$ be the 3D region obtained by the rotation of $U_k$ about the $z$-axis. Define $\widetilde{v}(r,\theta,z)=v(r,z)$ and $\widetilde{p}(r,\theta,z)=p(r,z)$ for $(r,\theta,z)\in\widetilde{U}_K$. Therefore, $\int_{\widetilde{U}_K}\widetilde{p}dx=\int_{\widetilde{U}_K}\widetilde{v}dx$. Namely, $\widetilde{p}$ is the $L^2$ projection of $\widetilde{v}$ on the constant function space over $\widetilde{U}_K$. Then, by Proposition 2.2 and the Poincaré inequality on $\widetilde{U}_K$, we have

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} \le Ch_K^{-1}\|\widetilde{v}-\widetilde{p}\|_{L^2(\widetilde{U}_K)} \le C|\widetilde{v}|_{H^1(\widetilde{U}_K)} \le C|v|_{H_1^1(U_K)}. \tag{3.38}$$

Combining (3.35) – (3.38), we have shown for $v \in H^1_{+,0}(\Omega)$,

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} \le C|v|_{H_1^1(\mathbb{U}_K)}. \tag{3.39}$$

Case II ($v \in H^1_{-,0}(\Omega)$). Following similar analysis as for Case I, it is straightforward to show that when [I] $U_K \cap \partial\Omega=\emptyset$ and when [II] $U_K\cap\Gamma\neq\emptyset$,

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} \le C|v|_{H_1^1(\mathbb{U}_K)}. \tag{3.40}$$

Therefore, we shall not repeat the calculations for these two situations. However, different analysis is needed for the patch $U_K$ satisfying [III] $U_K\cap\Gamma=\emptyset$ and $U_K\cap\{r=0\}\neq\emptyset$, which we present as follows. Choose $p\in V_h^-$, such that $p|_{U_K}=0$. Then, by the definition of the weighted space, we have

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} = h_K^{-1}\|v\|_{L_1^2(U_K)} \le C\|r^{-1}v\|_{L_1^2(U_K)}. \tag{3.41}$$

Combining (3.40) and (3.41), we have shown for $v\in H^1_{-,0}(\Omega)$,

$$h_K^{-1}\|v-p\|_{L_1^2(U_K)} \le C\|v\|_{H_-^1(\mathbb{U}_K)}. \tag{3.42}$$

Hence, by (3.33), (3.34), (3.39), and (3.42), we have completed the proof. □

Using the estimates derived above, we obtain the stability of the interpolation operator in weighted norms.

**Theorem 3.10.** *The interpolation operator $\mathcal{I}: \boldsymbol{V}\to\boldsymbol{V}_h$ (Definition 3.1) is stable. Namely,*

$$\|\mathcal{I}^+v\|_{H_+^1(\Omega)} \le C\|v\|_{H_+^1(\Omega)}, \qquad \forall v\in H^1_{+,0}(\Omega);$$
$$\|\mathcal{I}^-v\|_{H_-^1(\Omega)} \le C\|v\|_{H_-^1(\Omega)}, \qquad \forall v\in H^1_{-,0}(\Omega).$$

**Proof.** For $v\in H^1_{+,0}(\Omega)$, by (3.25), we have

$$\|\mathcal{I}^+v\|^2_{L_1^2(\Omega)} = \sum_{K\in\mathcal{T}_h}\|\mathcal{I}^+v\|^2_{L_1^2(K)} \le C\sum_{K\in\mathcal{T}_h}(h_K^2|v|^2_{H_1^1(U_K)}+\|v\|^2_{L_1^2(U_K)}) \le C\|v\|^2_{H_+^1(\Omega)}.$$

This, together with (3.31), leads to the first estimate of this theorem.

For $v\in H^1_{-,0}(\Omega)$, by (3.27), we have

$$\|r^{-1}\mathcal{I}^-v\|^2_{L_1^2(\Omega)} = \sum_{K\in\mathcal{T}_h}\|r^{-1}\mathcal{I}^-v\|^2_{L_1^2(K)} \le C\sum_{K\in\mathcal{T}_h}\|v\|^2_{H_-^1(U_K)} \le C\|v\|^2_{H_-^1(\Omega)}.$$

This, together with (3.32), leads to the second estimate of this theorem. □

## 4. The macroelement analysis

In this section, we present our main result (Theorem 4.9), a sufficient condition to verify the well-posedness (2.17) of the mixed finite element approximation of the ASP (2.16).

### 4.1. Macroelements

We start with the definition of the macroelement.

**Definition 4.1.** (Macroelements). Given a triangulation $\mathcal{T}_h$ consisting of triangles or quadrilaterals, as described in Section 2, a macroelement $M$ is a union of adjacent elements in $\mathcal{T}_h$. We say $M$ is *equivalent* to a reference macroelement $\hat{M}$ if there is a continuous and invertible mapping $F_M : \hat{M} \to M$, such that

1. $F_M(\hat{M}) = M$.
2. If $\hat{M} = \cup_{j \le J} \hat{K}_j$, where $\hat{K}_j$ are the elements that define $\hat{M}$, then $K_j = F_M(\hat{K}_j)$ are the elements in $M$.
3. $F_M|_{\hat{K}_j} = F_{K_j} \circ F^{-1}_{\hat{K}_j}$, where $F_{K_j}$ and $F_{\hat{K}_j}$ are the affine or bilinear mappings from the reference (triangular or square) element onto $K_j$ and $\hat{K}_j$, respectively.

Let $\mathcal{M}_h := \{M_k\}$ be the collection of all the macroelements. We further assume the following conditions.

(I) Each macroelement $M \in \mathcal{M}_h$ belongs to an equivalence class of macroelements $\mathcal{E}_{\hat{M}}$.
(II) The number of macroelement classes is finite, independent of $h$.
(III) For any element $K_j \subset M$, $\mathrm{diam}(M) \simeq \mathrm{diam}(K_j)$.
(IV) Each $K \in \mathcal{T}_h$ belongs to a finite number of macroelements, independent of $h$.

Recall the finite element spaces $\boldsymbol{V}_h$ and $P_h$ from Section 2. Then, on each macroelement $M \in \mathcal{M}_h$, we define the following local discrete spaces

$$\boldsymbol{V}_{0,M} := \{\boldsymbol{w} = \boldsymbol{v}|_M,\ \boldsymbol{v} \in \boldsymbol{V}_h,\ \boldsymbol{v}|_{\Omega \setminus M} = \boldsymbol{0}\}, \tag{4.1}$$

$$P_M := \{q = p|_M,\ p \in P_h\} \quad \text{and} \quad P_{0,M} := P_M \cap L^2_{1,0}(M). \tag{4.2}$$

For any $(\boldsymbol{v}, q) \in \boldsymbol{V}_{0,M} \times P_M$, define the bilinear forms on the local spaces

$$b(\boldsymbol{v}, q)_M := -\int_M (q\,\mathrm{div}_c \boldsymbol{v} + r^{-1} q v_r) r\,dr\,dz \quad \text{and} \quad b^s(\boldsymbol{v}, q)_M := -\int_M q\,\mathrm{div}_c \boldsymbol{v}\,dr\,dz.$$

For $\boldsymbol{v} \in \boldsymbol{V}_{0,M}$, we denote by $N_M$ and $N^s_M$ the local null-spaces of the linear mappings $b(\boldsymbol{v}, \cdot)_M : P_M \to \mathbb{R}$ and $b^s(\boldsymbol{v}, \cdot)_M : P_M \to \mathbb{R}$, respectively. Namely,

$$N_M := \{q \in P_M,\ b(\boldsymbol{v}, q)_M = 0,\ \forall \boldsymbol{v} \in \boldsymbol{V}_{0,M}\} \quad \text{and} \quad N^s_M := \{q \in P_M,\ b^s(\boldsymbol{v}, q)_M = 0,\ \forall \boldsymbol{v} \in \boldsymbol{V}_{0,M}\}. \tag{4.3}$$

We assume the following conditions on the spaces $N_M$ and $N^s_M$.

**Assumption 4.2.** For every $M \in \mathcal{M}_h$, each of the spaces $N_M$ and $N^s_M$ in (4.3) is one-dimensional and consists of the constant function on $M$.

**Remark 4.3.** Based on Definition 4.1, a macroelement is determined by the selected elements in a local patch of the mesh. The bilinear form $b(\cdot,\cdot)_M$ is the restriction of the global bilinear form $b(\cdot,\cdot)$ in (2.11) on the macroelement $M$. The bilinear form $b^s(\cdot,\cdot)_M$ can be regarded as the restriction of the non-symmetric part of the bilinear form from the usual 2D Stokes problem on $M$. For the usual 2D Stokes problem, only the condition on $N^s_M$ needs to be checked to ensure the global inf-sup condition [26]. For the ASP, we here need the conditions on both $N_M$ and $N^s_M$. Nevertheless, the conditions in Assumption 4.2 are local and relatively easy to verify in practice.

We first show that Assumption 4.2 implies the local inf-sup conditions for the bilinear forms $b(\cdot,\cdot)_M$ and $b^s(\cdot,\cdot)_M$ on the macroelement that touches the $z$-axis.

**Lemma 4.4.** *For each macroelement $M \in \mathcal{E}_{\hat{M}}$, suppose that one of its vertices is at the origin, and its diameter $h_M = 1$. Then, under Assumption 4.2, for any $q \in P_{0,M}$, the local inf-sup conditions hold*

$$\sup_{\boldsymbol{0} \ne \boldsymbol{v} \in \boldsymbol{V}_{0,M}} \frac{b(\boldsymbol{v}, q)_M}{\|\boldsymbol{v}\|_{\boldsymbol{V}_1(M)}} \ge \gamma_{1,\hat{M}} \|q\|_{L^2_1(M)}, \tag{4.4}$$

$$\sup_{\boldsymbol{0} \ne \boldsymbol{v} \in \boldsymbol{V}_{0,M}} \frac{b^s(\boldsymbol{v}, q)_M}{|\boldsymbol{v}|_{[H^1(M)]^2}} \ge \gamma_{2,\hat{M}} \|q\|_{L^2(M)}, \tag{4.5}$$

*where $\|\boldsymbol{v}\|^2_{\boldsymbol{V}_1(M)} = \|v_r\|^2_{H^1_-(M)} + |v_z|^2_{H^1_1(M)}$; and the constants $\gamma_{1,\hat{M}}, \gamma_{2,\hat{M}} > 0$ depend on the reference macroelement $\hat{M}$.*

**Proof.** The estimate (4.5) was given in Lemma 3.1 [26], since $b^s(\cdot,\cdot)$ coincides with the non-symmetric bilinear form in the usual Stokes problem. We here present the proof for (4.4).

Let $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_d)$ be the vertices of the elements in the reference macroelement $\hat{M}$, and let $(x_1, x_2, \cdots, x_d)$ be the set of corresponding vertices in $M$, where, without loss of generality, we assume $x_1 = (0, 0)$. Thus, $M$ is uniquely identified by its vertices via $x_i = F_M(\hat{x}_i)$, $1 \le i \le d$.

Define the constant $\gamma_{1,M}$

$$\gamma_{1,M} = \inf_{0 \ne q \in P_{0,M}} \sup_{\boldsymbol{0} \ne \boldsymbol{v} \in \boldsymbol{V}_{0,M}} \frac{b(\boldsymbol{v}, q)_M}{\|\boldsymbol{v}\|_{\boldsymbol{V}_1(M)} \|q\|_{L^2_1(M)}}. \tag{4.6}$$

Note that $\gamma_{1,M} > 0$ due to Assumption 4.2 and the spaces being finite-dimensional. If we consider the vertices of $M$ as a point $\mathbf{X} = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^{2d}$, then $\gamma_{1,M} = \gamma_{1,M}(\mathbf{X})$ is a continuous function of $\mathbf{X}$. Due to the shape regularity assumption, each element $K \subset M$ has a diameter $\simeq 1$. Therefore, the point $\mathbf{X}$ belongs to a compact set $D \in \mathbb{R}^{2d}$. Since the continuous function $\gamma_{1,M}(\mathbf{X}) > 0$ for any $\mathbf{X} \in D$, there is $\gamma_{1,\hat{M}} > 0$, such that $\gamma_{1,M} \geq \gamma_{1,\hat{M}}$ for any $M \in \mathcal{E}_{\hat{M}}$. This proves the estimate (4.4). □

Based on Lemma 4.4, we now derive the local inf-sup condition for the bilinear form $b(\cdot,\cdot)_M$ on arbitrary macroelements.

**Lemma 4.5.** *For a macroelement $M \in \mathcal{E}_{\hat{M}}$, under Assumption 4.2, we have*

$$\sup_{0\neq \boldsymbol{v}\in \boldsymbol{V}_{0,M}} \frac{b(\boldsymbol{v},q)_M}{\|\boldsymbol{v}\|_{V(M)}} \geq \gamma_{\hat{M}} \|q\|_{L^2_1(M)}, \quad \forall q \in P_{0,M}, \tag{4.7}$$

*where* $\|\boldsymbol{v}\|^2_{\boldsymbol{V}(M)} = \|v_r\|^2_{H^1_-(M)} + \|v_z\|^2_{H^1_+(M)}$, *and* $\gamma_{\hat{M}} > 0$ *is independent of* $h_M$.

**Proof.** We start with some arguments similar to the ones in Lemma 4.4. Here, we need to pay special attention to the weight $r$ in $b(\boldsymbol{v},q)_M$, which depends on the location of the macroelement.

For a macroelement $M \in \mathcal{M}_h$, define the constant $\gamma_M$

$$\gamma_M = \inf_{0\neq q\in P_{0,M}} \sup_{0\neq \boldsymbol{v}\in \boldsymbol{V}_{0,M}} \frac{b(\boldsymbol{v},q)_M}{\|\boldsymbol{v}\|_{V(M)}\|q\|_{L^2_1(M)}}. \tag{4.8}$$

With the spaces involved being finite-dimensional and Assumption 4.2, we have $\gamma_M > 0$. We shall show that there exists $\gamma_{\hat{M}} > 0$, such that $\gamma_M \geq \gamma_{\hat{M}}$ for all the macroelements in the same equivalence class $M \in \mathcal{E}_{\hat{M}}$.

Let $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_d)$ be the vertices of the triangles or quadrilaterals in the reference macroelement $\hat{M}$, and let $(x_1, x_2, \cdots, x_d)$ be the set of corresponding vertices in $M$. Without loss of generality, we assume $x_1 = (r_1, z_1)$ is the closest vertex to the $z$-axis.

A simple translation and dilation

$$s = h_M^{-1}(r - r_1), \qquad t = h_M^{-1}(z - z_1),$$

of the $rz$ system allows us to place $x_1$ at the origin of the new $st$ system, and translate $M$ into

$$M' = \{(s,t),\ s = h_M^{-1}(r - r_1),\ t = h_M^{-1}(z - z_1),\ (r,z) \in M\}, \tag{4.9}$$

whose diameter $h_{M'} = 1$. For any function $w$ on $M$, define its dilation $w'$ on $M'$ by

$$w'(s,t) = w(r,z).$$

Let $x'_i$, $1 \leq i \leq d$, be the $i$th vertex of $M'$ corresponding to the vertex $x_i$ of $M$ after the dilation.

Then, by the scaling argument, we have

$$\begin{aligned} b(\boldsymbol{v},q)_M &= -\int_M (q\partial_r v_r + q\partial_z v_z + r^{-1} q v_r) r\,dr\,dz \\ &= -h_M^2 \int_{M'} (q'\partial_s v'_r + q'\partial_t v'_z + s^{-1} q' v'_r) s\,ds\,dt - r_1 h_M \int_{M'} (q'\partial_s v'_r + q'\partial_t v'_z)\,ds\,dt \\ &= h_M^2 b(\boldsymbol{v}', q')_{M'} + r_1 h_M b^s(\boldsymbol{v}', q')_{M'}. \end{aligned} \tag{4.10}$$

We consider the following two cases for the estimates of $\gamma_M$.

Case I ($M \cap \{r = 0\} = \emptyset$). We first simplify the denominator in (4.8). For $\boldsymbol{v} = (v_r, v_z) \in \boldsymbol{V}_{0,M}$, let $\boldsymbol{w} = (w_r, w_z) \in \boldsymbol{V}_h$ be the extension of $\boldsymbol{v}$ by zero outside of $M$ and $\boldsymbol{v} = \boldsymbol{w}|_M$. Therefore, $w_r \in H^1_0(M) \subset H^1_0(\Omega)$. By the estimate (3.7), we have

$$\begin{aligned} \int_M r^{-1} v_r^2\,dr\,dz &\leq C r_1 \int_M r^{-2} v_r^2\,dr\,dz = C r_1 \int_\Omega r^{-2} w_r^2\,dr\,dz \\ &\leq C r_1 |w_r|^2_{H^1(\Omega)} = C r_1 |v_r|^2_{H^1(M)}. \end{aligned}$$

Meanwhile, since $w_z \in H^1_0(M) \subset H^1_0(\Omega)$, by the Poincaré inequality, we have

$$\begin{aligned} \int_M v_z^2 r\,dr\,dz &\leq C r_1 \int_M v_z^2\,dr\,dz = C r_1 \int_\Omega w_z^2\,dr\,dz \\ &\leq C r_1 |w_z|^2_{H^1(\Omega)} = C r_1 |v_z|^2_{H^1(M)}. \end{aligned}$$

In addition, we have

$$\int_M q^2 r\,dr\,dz \leq C r_1 \int_M q^2\,dr\,dz.$$

Note that the constant $C$ in the estimates above depends on $\hat{M}$ and the shape regularity of the mesh. Therefore, by the definition of the norms and the scaling argument, we have

$$\|\boldsymbol{v}\|_{V(M)}\|q\|_{L_1^2(M)} \simeq r_1|\boldsymbol{v}|_{[H^1(M)]^2}\|q\|_{L^2(M)} = r_1 h_M|\boldsymbol{v}'|_{[H^1(M')]^2}\|q'\|_{L^2(M')}. \tag{4.11}$$

Define $\lambda_M := h_M/r_1$. Thus, by (4.8), (4.10), and (4.11), we have

$$\gamma_M \simeq \gamma_{M'} := \inf_{0\neq q'\in P_{0,M'}} \sup_{\boldsymbol{0}\neq \boldsymbol{v}'\in \boldsymbol{V}_{0,M'}} \frac{\lambda_M b(\boldsymbol{v}',q')_{M'} + b^s(\boldsymbol{v}',q')_{M'}}{|\boldsymbol{v}'|_{[H^1(M')]^2}\|q'\|_{L^2(M')}}. \tag{4.12}$$

Let $\boldsymbol{w}'$ be the extension of $\boldsymbol{v}'$ by zero outside of $M'$ and let $M^* = [0,1]\times[-1,1]$. It is clear that $M' \subset M^*$. Then, by Hölder's inequality, $0 \le s \le 1$, and (3.7), we have

$$\begin{aligned}
|b(\boldsymbol{v}',q')_{M'}| &= |\int_{M'} (q'\partial_s v_r' + q\partial_t v_z' + s^{-1}q'v_r')s\,ds\,dt| \\
&\le \|q'\|_{L^2(M')}|\boldsymbol{v}'|_{[H^1(M')]^2} + \|q'\|_{L^2(M')}\|s^{-1}v_r'\|_{L^2(M')} \\
&= \|q'\|_{L^2(M')}|\boldsymbol{v}'|_{[H^1(M')]^2} + \|q'\|_{L^2(M')}\|s^{-1}w_r'\|_{L^2(M^*)} \\
&\le \|q'\|_{L^2(M')}|\boldsymbol{v}'|_{[H^1(M')]^2} + c^*\|q'\|_{L^2(M')}|w_r'|_{H^1(M^*)} \\
&\le (1+c^*)\|q'\|_{L^2(M')}|\boldsymbol{v}'|_{[H^1(M')]^2},
\end{aligned} \tag{4.13}$$

where $c^*$ depends on $M^*$, not on $M'$. Therefore, when $\lambda_M < \gamma_{2,\hat{M}}/(2(1+c^*))$, where $\gamma_{2,\hat{M}}$ is the parameter given in (4.5). Then, by (4.13) and (4.5), we have

$$\gamma_{M'} = \inf_{0\neq q'\in P_{0,M'}} \sup_{\boldsymbol{0}\neq \boldsymbol{v}'\in \boldsymbol{V}_{0,M'}} \frac{\lambda_M b(\boldsymbol{v}',q')_{M'} + b^s(\boldsymbol{v}',q')_{M'}}{|\boldsymbol{v}'|_{[H^1(M')]^2}\|q'\|_{L^2(M')}} \ge \frac{\gamma_{2,\hat{M}}}{2}. \tag{4.14}$$

When $\lambda_M \ge \gamma_{2,\hat{M}}/(2(1+c^*))$, note that $\lambda_M = h_M/r_1 \le c_{\mathcal{T}}$, where $c_{\mathcal{T}} > 0$, depending on the shape regularity of the mesh, is the global upper bound of $\lambda_M$ for any $M \in \mathcal{M}_h$. By (4.12) and (4.8), $\gamma_{M'} \simeq \gamma_M > 0$. If we consider the vertices of $M'$ as a point $\mathbf{X}' = (x_1', x_2', \cdots, x_d') \in \mathbb{R}^{2d}$, it is clear that $\gamma_{M'}$ is a continuous function of $\mathbf{X}'$ and $\lambda_M$, $\gamma_{M'} = \gamma_{M'}(\mathbf{X}', \lambda_M)$. By the shape regularity of the mesh, $h_{M'} = 1$, and by the fact that $\gamma_{2,\hat{M}}/(2(1+c^*)) \le \lambda_M \le c_{\mathcal{T}}$, $(\mathbf{X}', \lambda_M)$ belongs to a compact set $D \in \mathbb{R}^{2d+1}$. Since $\gamma_{M'}(\mathbf{X}', \lambda_M) > 0$ for any $(\mathbf{X}', \lambda_M) \in D$, there is $\gamma_{\hat{M}}' > 0$, such that for $\gamma_{2,\hat{M}}/(2(1+c^*)) \le \lambda_M \le c_{\mathcal{T}}$,

$$\gamma_{M'} \ge \gamma_{\hat{M}}'. \tag{4.15}$$

Hence, by (4.12), (4.14), and (4.15), we conclude that there exists a constant $\gamma_{\hat{M}}^* > 0$, independent of $h_M$, such that when $M \cap \{r=0\} = \emptyset$,

$$\gamma_M \ge \gamma_{\hat{M}}^*. \tag{4.16}$$

Case II ($M \cap \{r=0\} \neq \emptyset$). Recall $M'$ from (4.9). Note that $r_1 = 0$ in this case. For $\boldsymbol{v} \in \boldsymbol{V}_{0,M}$, let $\boldsymbol{w} = (w_r, w_z) \in \boldsymbol{V}_h$ be the extension of $\boldsymbol{v}$ by zero outside of $M$ and $\boldsymbol{v} = \boldsymbol{w}|_M$. Then, $w_z \in H^1_{+,0}(\Omega)$. By Proposition 2.5, we first have

$$\|v_z\|^2_{H^1_+(M)} = \|w_z\|^2_{H^1_+(\Omega)} \le C|w_z|^2_{H^1_1(\Omega)} = C|v_z|^2_{H^1_1(M)},$$

where $C$ is independent of $M$. Then, using the scaling argument, we have

$$\begin{aligned}
\|\boldsymbol{v}\|^2_{V(M)} &\simeq |v_r|^2_{H^1_1(M)} + \int_M r^{-1}v_r^2\,dr\,dz + |v_z|^2_{H^1_1(M)} \\
&= h_M(|\boldsymbol{v}'|^2_{[H^1_1(M')]^2} + \int_{M'} s^{-1}(v_r')^2\,ds\,dt) = h_M\|\boldsymbol{v}'\|^2_{V_1(M')},
\end{aligned}$$

where $\|\cdot\|_{V_1(M')}$ is the norm defined in Lemma 4.4 and the constant involved in the estimates is independent of $M'$. Similarly, we have

$$\|q\|^2_{L^2_1(M)} = \int_M q^2 r\,dr\,dz = h_M^3\|q'\|^2_{L^2_1(M')}.$$

Therefore,

$$\|\boldsymbol{v}\|_{V(M)}\|q\|_{L^2_1(M)} \simeq h_M^2\|\boldsymbol{v}'\|_{V_1(M')}\|q'\|_{L^2_1(M')}. \tag{4.17}$$

Then, by (4.8), (4.10), (4.17), and (4.4), we have

$$\gamma_M \simeq \inf_{0\neq q\in P_{0,M'}} \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \boldsymbol{V}_{0,M'}} \frac{b(\boldsymbol{v}',q')_{M'}}{\|\boldsymbol{v}'\|_{V_1(M')}\|q'\|_{L^2_1(M')}} \ge \gamma_{1,\hat{M}}. \tag{4.18}$$

The proof of (4.7) is thus completed by (4.16) and (4.18). □

**Remark 4.6.** Although the form $b^s(\cdot,\cdot)_M$ does not appear in the estimate (4.7), the inequality (4.5) is an important ingredient of the proof involved, due to the decomposition (4.10). Thus, based on the local conditions in Assumption 4.2, we have obtained the local inf-sup condition in Lemma 4.5 for any macroelement $M \in \mathcal{M}_h$. This will help to achieve the global inf-sup condition in the next subsection.

### 4.2. The inf-sup condition

Now, we present some additional estimates that will enable us to derive the global inf-sup condition (2.17) from the local estimate in (4.7). First, we define the piecewise constant space

$$Q_h := \{v \in L^2_{1,0}(\Omega),\ v|_K \in \mathbb{R},\ \forall K \in \mathcal{T}_h\}.$$

Recall the discrete pressure space $P_h \subset L^2_{1,0}(\Omega)$. Let $\Pi_M : P_h \to \mathbb{R}$ and $\Pi_K : P_h \to \mathbb{R}$ be the $L^2_1$-projection operators onto the constant space over the macroelement $M$ and the element $K$, respectively, such that

$$\int_M (\Pi_M q) r dr dz = \int_M q r dr dz, \quad \forall q \in P_h,\ M \in \mathcal{M}_h,$$
$$\int_K (\Pi_K q) r dr dz = \int_K q r dr dz, \quad \forall q \in P_h,\ K \in \mathcal{T}_h.$$

Define the operator $\Pi_Q : P_h \to Q_h$, such that for $q \in P_h$, $\Pi_Q q|_K := \Pi_K q$. Recall the space $\boldsymbol{V} = H^1_{-,0}(\Omega) \times H^1_{+,0}(\Omega)$. Let $I$ be the identity operator. Then, we have the following estimate.

**Lemma 4.7.** *Under Assumption 4.2, for any $q \in P_h$, there is $\boldsymbol{v}_1 \in \boldsymbol{V}_h$, such that*

$$\|\boldsymbol{v}_1\|^2_{\boldsymbol{V}} \le C_0 \sum_{M\in\mathcal{M}_h} \|(I-\Pi_M)q\|^2_{L^2_1(M)} \tag{4.19}$$

*and*

$$b(\boldsymbol{v}_1, q) \ge C_1 \|(I-\Pi_Q)q\|^2_{L^2_1(\Omega)}, \tag{4.20}$$

*where $C_0, C_1 > 0$ are independent of the mesh size.*

**Proof.** For $q \in P_h$, on each $M \in \mathcal{M}_h$, let $q_M := q|_M$. Then, by the definition of $\Pi_M$,

$$(I-\Pi_M)q_M \in P_{0,M}.$$

By Assumption 4.2 and the inf-sup condition (4.7), on each macroelement $M$, there exists $\boldsymbol{v}_M \in \boldsymbol{V}_{0,M}$, such that

$$\begin{aligned} b(\boldsymbol{v}_M, q_M)_M &= b(\boldsymbol{v}_M, (I-\Pi_M)q_M)_M = \|(I-\Pi_M)q_M\|^2_{L^2_1(M)}, \\ \|\boldsymbol{v}_M\|_{V(M)} &\le C\|(I-\Pi_M)q_M\|_{L^2_1(M)} = C\|(I-\Pi_M)q\|_{L^2_1(M)}, \end{aligned} \tag{4.21}$$

where $C$ depends on $\gamma_{\hat{M}}$ in (4.7). For each $\boldsymbol{v}_M$, let $\boldsymbol{w}_M \in \boldsymbol{V}_h$ be such that $\boldsymbol{v}_M = \boldsymbol{w}_M|_M$ and $\boldsymbol{w}_M|_{\Omega\setminus M} = 0$. Define $\boldsymbol{v}_1 := \sum_M \boldsymbol{w}_M \in \boldsymbol{V}_h$. For $K \in \mathcal{T}_h$, let $\|\cdot\|^2_{\boldsymbol{V}(K)} := \|\cdot\|^2_{H^1_-(K)} + \|\cdot\|^2_{H^1_+(K)}$. Thus, since each $K$ belongs to a finite number of macroelements, by the triangle inequality, (4.21), and (II) in Definition 4.1, we have

$$\begin{aligned} \|\boldsymbol{v}_1\|^2_{\boldsymbol{V}} &= \sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}_1\|^2_{\boldsymbol{V}(K)} = \sum_{K\in\mathcal{T}_h} \|\sum_{M\in\mathcal{M}_h, K\cap M\neq\emptyset} \boldsymbol{w}_M\|^2_{\boldsymbol{V}(K)} \\ &= \sum_{K\in\mathcal{T}_h} \|\sum_{M\in\mathcal{M}_h, K\cap M\neq\emptyset} \boldsymbol{v}_M\|^2_{\boldsymbol{V}(K)} \le C \sum_{K\in\mathcal{T}_h} (\sum_{M\in\mathcal{M}_h, K\cap M\neq\emptyset} \|\boldsymbol{v}_M\|^2_{\boldsymbol{V}(K)}) \\ &\le C \sum_{M\in\mathcal{M}_h} \|\boldsymbol{v}_M\|^2_{\boldsymbol{V}(M)} \le C_0 \sum_{M\in\mathcal{M}_h} \|(I-\Pi_M)q\|^2_{L^2_1(M)}. \end{aligned}$$

This proves (4.19). In addition, by the condition (IV) in Definition 4.1, we have

$$\begin{aligned} b(\boldsymbol{v}_1, q) &= \sum_{M\in\mathcal{M}_h} b(\boldsymbol{w}_M, q) = \sum_{M\in\mathcal{M}_h} b(\boldsymbol{v}_M, q_M)_M = \sum_{M\in\mathcal{M}_h} \|(I-\Pi_M)q_M\|^2_{L^2_1(M)} \\ &\ge \sum_{M\in\mathcal{M}_h} \sum_{K\subset M} \|(I-\Pi_K)q_M\|^2_{L^2_1(K)} \ge C_1 \|(I-\Pi_Q)q\|^2_{L^2_1(\Omega)}, \end{aligned}$$

where we used the fact that $\|(I-\Pi_M)q_M\|^2_{L^2_1(M)} = \sum_{K\subset M} \|(I-\Pi_M)q_M\|^2_{L^2_1(K)} \ge \sum_{K\subset M} \|(I-\Pi_K)q_M\|^2_{L^2_1(K)}$. Namely, $\Pi_K q_M$ is the best constant approximation of $q_M$ in $L^2_1(K)$. This proves (4.20).

Hence, the proof is completed. □

We shall also need the following construction for a specific function in $\boldsymbol{V}_h$.

**Lemma 4.8.** *Under Assumption 4.2, for any $q \in P_h$, there is $\boldsymbol{v}_2 \in \boldsymbol{V}_h$, such that*

$$b(\boldsymbol{v}_2, \Pi_Q q) = \|\Pi_Q q\|^2_{L^2_1(\Omega)}, \tag{4.22}$$

*where*

$$\|\boldsymbol{v}_2\|_V \le C_2 \|\Pi_Q q\|_{L^2_1(\Omega)}, \tag{4.23}$$

*and $C_2$ is independent of the mesh size.*

**Proof.** Let $\widetilde{q}$ be the axisymmetric function defined in the 3D domain $\widetilde{\Omega}$, such that $\widetilde{q}(r,\theta,z) = (\Pi_Q q)(r,z)$. Therefore, $\int_{\widetilde{\Omega}} \widetilde{q}\,dx\,dy\,dz = 2\pi \int_\Omega (\Pi_Q q) r\,dr\,dz = 0$. Hence, by Lemma 2.7, there exists $\widetilde{\boldsymbol{w}} = (\widetilde{w}_r, \widetilde{w}_\theta, \widetilde{w}_z) \in \widetilde{\mathbf{H}}^1_0(\widetilde{\Omega})$, such that

$$-\mathrm{div}\widetilde{\boldsymbol{w}} = \widetilde{q}$$

and $\|\widetilde{\boldsymbol{w}}\|_{[H^1(\widetilde{\Omega})]^3} \le C\|\widetilde{q}\|_{L^2(\widetilde{\Omega})}$. Recall $\widetilde{w}_r, \widetilde{w}_\theta, \widetilde{w}_z$ are all axisymmetric functions. In view of Proposition 2.2, let $\boldsymbol{w} = (w_r, w_z) \in \boldsymbol{V}$, where $w_r$ and $w_z$ are the traces of $\widetilde{w}_r$ and $\widetilde{w}_z$ on $\Omega$ as defined in (2.5). Then, a direct calculation leads to

$$\begin{aligned} b(\boldsymbol{w}, \Pi_Q q) &= -\int_\Omega (\mathrm{div}_c \boldsymbol{w} + r^{-1} w_r) \Pi_Q q r\,dr\,dz \\ &= -\frac{1}{2\pi}\int_{\widetilde{\Omega}} \widetilde{q}\,\mathrm{div}\widetilde{\boldsymbol{w}}\,dx\,dy\,dz = \frac{1}{2\pi}\|\widetilde{q}\|^2_{L^2(\widetilde{\Omega})} = \|\Pi_Q q\|^2_{L^2_1(\Omega)}. \end{aligned} \tag{4.24}$$

In addition, by Proposition 2.2, Lemma 2.7, and $\widetilde{q}(r,\theta,z) = (\Pi_Q q)(r,z)$, we have

$$\|\boldsymbol{w}\|_V \le C\|\widetilde{\boldsymbol{w}}\|_{[H^1(\widetilde{\Omega})]^3} \le C\|\widetilde{q}\|_{L^2(\widetilde{\Omega})} \le C\|\Pi_Q q\|_{L^2_1(\Omega)}. \tag{4.25}$$

Let $\boldsymbol{v}_2 = \mathcal{I}\boldsymbol{w} \in \boldsymbol{V}_h$, where $\mathcal{I}$ is the interpolation operator in Definition 3.1. Then, by (3.6), Theorem 3.10, (4.24), and (4.25), we have

$$b(\boldsymbol{v}_2, \Pi_Q q) = b(\boldsymbol{w}, \Pi_Q q) = \|\Pi_Q q\|^2_{L^2_1(\Omega)} \quad \text{and} \quad \|\boldsymbol{v}_2\|_V \le C\|\boldsymbol{w}\|_V \le C_2\|\Pi_Q q\|_{L^2_1(\Omega)},$$

which completes the proof. □

Lemma 4.7 and Lemma 4.8 provide technical ingredients to obtain our main result regarding the global well-posedness of the finite element approximation (2.16) of the ASP.

**Theorem 4.9.** *Under Assumption 4.2, the inf-sup condition (2.17) holds.*

**Proof.** It suffices to show that there exists $\gamma > 0$, independent of $h$, such that

$$\sup_{0 \ne \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} \ge \gamma \|q\|_{L^2_1(\Omega)}, \quad \forall q \in P_h, \tag{4.26}$$

where $b(\cdot,\cdot)$ is given in (2.11). For $q \in P_h$, we define $\boldsymbol{v} = \boldsymbol{v}_1 + \delta\boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the functions in Lemmas 4.7 and 4.8, respectively, and $\delta > 0$ will be determined later. Then, by (4.20), (4.22), Hölder's inequality, (4.23), and Young's inequality, we have

$$\begin{aligned} b(\boldsymbol{v}, q) &= b(\boldsymbol{v}_1, q) + \delta b(\boldsymbol{v}_2, \Pi_Q q) + \delta b(\boldsymbol{v}_2, (I - \Pi_Q) q) \\ &\ge C_1 \|(I - \Pi_Q) q\|^2_{L^2_1(\Omega)} + \delta \|\Pi_Q q\|^2_{L^2_1(\Omega)} - \delta \|\boldsymbol{v}_2\|_V \|(I - \Pi_Q) q\|_{L^2_1(\Omega)} \\ &\ge (C_1 - \frac{\delta C_2^2}{2}) \|(I - \Pi_Q) q\|^2_{L^2_1(\Omega)} + \frac{\delta}{2}\|\Pi_Q q\|^2_{L^2_1(\Omega)}. \end{aligned}$$

Choosing $\delta = 2C_1(C_2^2 + 1)^{-1}$, by the triangle inequality, we therefore have

$$b(\boldsymbol{v}, q) \ge C_1 (C_2^2 + 1)^{-1} \|q\|^2_{L^2_1(\Omega)}. \tag{4.27}$$

In addition, by (4.19), (4.23), the definitions of the projection operators, and (IV) in Definition 4.1, we have

$$\begin{aligned} \|\boldsymbol{v}\|^2_V \le C(\|\boldsymbol{v}_1\|^2_V + \|\boldsymbol{v}_2\|^2_V) &\le C(\sum_{M \in \mathcal{M}_h} \|(I - \Pi_M) q\|^2_{L^2_1(M)} + \|\Pi_Q q\|^2_{L^2_1(\Omega)}) \\ &\le C(\sum_{M \in \mathcal{M}_h} \|q\|^2_{L^2_1(M)} + \|q\|^2_{L^2_1(\Omega)}) \le C\|q\|^2_{L^2_1(\Omega)}. \end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28), we have proved the estimate (4.26), and hence the inf-sup condition (2.17). □

**Remark 4.10.** We have shown that in order to achieve the inf-sup condition (2.17) for the mixed finite element approximation of the ASP, it is sufficient to have the local macroelement conditions in Assumption 4.2. In addition, our analysis allows overlapping macroelements consisting of triangles or quadrilaterals (Definition 4.1), which gives more flexibility in verifying the local conditions. We expect this result, together with other intermediate finite element estimates obtained in this paper, to be useful in developing new mixed methods and theories for the ASP.

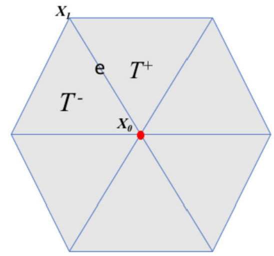


**Fig. 2.** A macroelement $M \in \mathcal{M}_h$ associated to an interior node $x_0$ in the mesh $\mathcal{T}_h$.

### 4.3. New stable mixed finite elements

We here present a number of stable mixed finite element methods for the ASP. Using the condition in Assumption 4.2 and Theorem 4.9, we can not only verify existing stable elements, but also obtain new elements for the numerical approximation. We remark that in Assumption 4.2, the condition on $N_M^s$ is the same as the macroelement condition for the standard Stokes problem [26], which has been shown to hold for several mixed methods, such as the Hood-Taylor elements and the $Q_2 - P_1$ element (see [7] and references therein). On the other hand, the condition on $N_M$ is an extra constraint for the ASP. Therefore, one can see that the stability of the method for the ASP is more stringent. Our theory also suggests that the stable mixed finite element for the ASP should be sought from stable mixed methods for standard 2D Stokes equations. One example for which both conditions are validated is the general Hood-Taylor elements $[P_{m+1}]^2 \times P_m$ with $m \geq 1$ [14,19]. In the rest of this section, we derive new stable elements that are locally mass conserving. We begin with the following definition.

**Definition 4.11** *(Local and global conservation).* The velocity field $\mathbf{u}$ is said to be locally and globally conservative if the following holds true:

$$\int_{\widetilde{T}} \widetilde{\nabla} \cdot \widetilde{\mathbf{u}}\, dx = 0, \quad \forall T \in \mathcal{T}_h, \tag{4.29}$$

where $\widetilde{T}$ is the Toroid of $T$, the axisymmetric extension of the 2D element $T$ in the azimuth direction.

Note that the above definition has been stated in terms of normal flux for the discontinuous Galerkin finite element methods. The local and global conservation holds if the pressure space contains piecewise constant functions, which has been noted to be crucial in obtaining reliable computational results. See [18] and references therein.

#### 4.3.1. Augmented Hood-Taylor elements

Consider the Hood-Taylor elements on triangular (2D) or on tetrahedral (3D) meshes: the continuous vector piecewise polynomial $P_{m+1}$ space for the velocity and the continuous piecewise polynomial $P_m$ space for the pressure. In a recent work [8], it was shown that for the usual Stokes problem, when the Hood-Taylor pressure space is augmented with the discontinuous piecewise constant functions, the resulting elements are stable in 2D ($m \geq 1$) and in 3D ($m \geq 2$). In particular, it was shown that for these augmented Hood-Taylor elements, $N_M^s$ (see Assumption 4.2) is one-dimensional and consists of constant functions on each macroelement $M$ in the mesh. In the text below, we shall prove these elements are also stable for solving the ASP. According to Theorem 4.9, we establish this stability by showing that the condition in Assumption 4.2 also holds on $N_M$, since the same condition has been verified on $N_M^s$ in [8].

Let $\mathcal{T}_h$ be a triangulation of the domain $\Omega$ consisting of shape regular triangles. For $m \geq 1$, we consider the following augmented Hood-Taylor elements solving the ASP,

$$\begin{aligned} \boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{v}|_T \in [P_{m+1}(T)]^2,\ \forall T \in \mathcal{T}_h\}, \\ P_h &:= \{p \in L^2_{1,0}(\Omega),\ p = p_m + p_0,\ p_m \in C(\Omega),\ p_m|_T \in P_m(T),\ p_0|_T \in P_0(T),\ \forall T \in \mathcal{T}_h\}. \end{aligned}$$

Here we use $T$ instead of $K$ to denote a triangle. Let $M \in \mathcal{M}_h$ be the macroelement associated to an interior node, such that $M$ is the collection of all the triangles attached to the interior node (Fig. 2). Note that these augmented Hood-Taylor elements are locally conservative for the ASP, which is a desired feature in practical computations.

**Lemma 4.12.** *Suppose that every triangle $T \in \mathcal{T}_h$ has at least one vertex in the interior of $\Omega$. Define $\mathcal{M}_h$ by grouping together, for each internal vertex $x_0$, those triangles that touch $x_0$. Then for every $M \in \mathcal{M}_h$, $N_M$ is one-dimensional, consisting of the constant function in $M$ for $m \geq 1$.*

**Proof.** Let $M$ be a macroelement associated to the internal vertex $x_0$ as shown in Fig. 2. Recall the local spaces $\boldsymbol{V}_{0,M}$ and $P_M$ in (4.1) – (4.2). Let $q \in N_M$, namely,

$$0 = \int_M q\left(\mathrm{div}_c \boldsymbol{v} + r^{-1} v_r\right) r\,dr\,dz = \frac{1}{2\pi}\int_{\widetilde{M}} \widetilde{q}\nabla \cdot \widetilde{\boldsymbol{v}}\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} = (v_r, v_z) \in \boldsymbol{V}_{0,M}, \tag{4.30}$$

where $\widetilde{q}$, $\widetilde{\boldsymbol{v}}$, and $\widetilde{M}$ are axisymmetric extensions of $q$, $\boldsymbol{v}$ and $M$,

$$\begin{aligned} &\widetilde{q}(r,\theta,z) = q(r,z), \qquad \widetilde{\boldsymbol{v}} = (\widetilde{v}_r, 0, \widetilde{v}_z), \\ &\qquad \widetilde{M} = M \times [-\pi, \pi). \end{aligned}$$

We also denote by $\widetilde{c} = c$ a constant function. Since $q \in P_M$, it is of the following form:

$$q = q_m + q_0,$$

where $q_m \in C(M), q_m|_T \in P_m(T)$ and $q_0|_T \in P_0(T)$, for any $T \subset M$. At this stage, it is crucial to observe that for every edge $e$ of $T \in M$ that contains $x_0$ as an end point, as shown in Fig. 2, it holds

$$\partial_e q = \partial_e q_m,$$

where $\partial_e q$ is the directional derivative of $q$ along the direction of $e$. Using this fact and Lemma 2.8 in [19], we obtain that $q_m$ has to be a constant in $M$. Namely, $q$ is a piecewise constant function in $M$. Now, we show that $q$ is a constant in $M$.

We let $T^+$ and $T^-$ be any two neighboring triangles in $M$ that share an edge $e$ (Fig. 2). We denote by $q^\pm$ the constant value of $q$ restricted to $T^\pm$, respectively. Our goal is to show that $q^+ = q^-$. We consider two cases: (I) the common edge $e$ is not parallel to the $z$-axis, and (II) the common edge $e$ is parallel to the $z$-axis. For case (I), we define $\boldsymbol{v}$ in the following way:

$$\boldsymbol{v}|_{T^\pm} = (0, \lambda_1^\pm \lambda_0^\pm),$$
$$\boldsymbol{v}|_T = \boldsymbol{0}, \quad \text{if } T \in \mathcal{T}_h,\ T \neq T^\pm,$$

where $\lambda_1^\pm$ and $\lambda_0^\pm$ are the linear basis functions in $T^\pm$, which are one at $x_1$ and $x_0$, respectively. Note that these functions are continuous across $e$. Thus, let $\lambda_i^e = \lambda_i^+|_e = \lambda_i^-|_e$ for $i = 0, 1$. Clearly, $\boldsymbol{v}$ is continuous piecewise quadratic, and $\boldsymbol{v} = \boldsymbol{0}$ on $\partial M$. Let $\mathbf{n} = (n_r, n_z)$ be the unit normal vector to $e$ from $T^+$ to $T^-$. Let $\widetilde{e} = e \times [-\pi, \pi)$ and let $\widetilde{\mathbf{n}} = (n_r, 0, n_z)$ be the axisymmetric extension of $\mathbf{n}$, the unit normal vector to $\widetilde{e}$ from $\widetilde{T}^+$ to $\widetilde{T}^-$ ($\widetilde{T}^\pm = T^\pm \times [-\pi, \pi)$). Then, by (4.30), we have

$$\begin{aligned}
0 &= \int_{\widetilde{M}} \widetilde{q} \nabla \cdot \widetilde{\boldsymbol{v}}\, d\boldsymbol{x} = q^+ \int_{\widetilde{T}^+} \nabla \cdot \widetilde{\boldsymbol{v}}\, d\boldsymbol{x} + q^- \int_{\widetilde{T}^-} \nabla \cdot \widetilde{\boldsymbol{v}}\, d\boldsymbol{x} \\
&= (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\boldsymbol{v}} \cdot \widetilde{\mathbf{n}}\, d\mathbf{s} = (q^+ - q^-) \int_{\widetilde{e}} (\widetilde{v}_r \cos\theta, \widetilde{v}_r \sin\theta, \widetilde{v}_z) \cdot (n_r \cos\theta, n_r \sin\theta, n_z)\, d\mathbf{s} \\
&= (q^+ - q^-) \int_{\widetilde{e}} (0, 0, \widetilde{v}_z) \cdot (n_r \cos\theta, n_r \sin\theta, n_z)\, d\mathbf{s} = (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\lambda}_1^e \widetilde{\lambda}_0^e n_z\, d\mathbf{s}.
\end{aligned}$$

Note that $\lambda_1^e \lambda_0^e$ is positive on $e$. Since $e$ is not parallel to the $z$-axis, we have $n_z \neq 0$ and therefore $\int_{\widetilde{e}} \widetilde{\lambda}_1^e \widetilde{\lambda}_0^e n_z\, d\mathbf{s} \neq 0$. Hence, we obtain that $q^+ = q^-$. For case (II), we define $\boldsymbol{v}$ in the following way:

$$\boldsymbol{v}|_{T^\pm} = (\lambda_1^\pm \lambda_0^\pm, 0),$$
$$\boldsymbol{v}|_T = 0, \quad \text{if } T \in \mathcal{T}_h,\ T \neq T^\pm.$$

Then, by similar arguments, we obtain that

$$0 = (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\boldsymbol{v}} \cdot \widetilde{\mathbf{n}}\, d\mathbf{s} = (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\lambda}_1^e \widetilde{\lambda}_0^e n_r d\mathbf{s}.$$

Note that $n_r \neq 0$ since $e$ is parallel to the $z$-axis. Using the similar argument, we have $q^+ = q^-$. This process can be applied to other edges containing $x_0$. Consequently, we arrive at the conclusion that $q$ is a constant function in $M$. This completes the proof. □

### 4.3.2. *Augmented Bercovier-Pironneau elements*

We study another conservative mixed method. Consider the augmented Bercovier-Pironneau elements for the ASP

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{v}|_T \in [P_1(T)]^2,\ \forall T \in \mathcal{T}_{h/2}\} \tag{4.31}$$
$$P_h := \{q \in L^2_{1,0}(\Omega),\ p = p_1 + p_0,\ p_1 \in C(\Omega),\ p_1|_T \in P_1(T),\ p_0|_T \in P_0(T),\ \forall T \in \mathcal{T}_h\}, \tag{4.32}$$

where $\mathcal{T}_{h/2}$ is the triangulation of $\Omega$ obtained by joining the midpoints of the edges of the triangles of $\mathcal{T}_h$. Note that this method has been shown to satisfy the condition (Assumption 4.2) on $N_M^s$ for the usual 2D Stokes equations [8]. Therefore, based on Theorem 4.9, it suffices to establish the condition on $N_M$ for the stability of the method solving the ASP.

**Lemma 4.13.** *Suppose that every triangle $T \in \mathcal{T}_h$ has at least one vertex in the interior of $\Omega$. Define $\mathcal{M}_h$ by grouping together, for each internal vertex $x_0$, those triangles that touch $x_0$. Then for every $M \in \mathcal{M}_h$, $N_M$ is one-dimensional, consisting of the constant function in $M$.*

**Proof.** As shown in Fig. 2, given an internal node $x_0$ and the macroelement $M$, we consider two triangles $T^+$ and $T^-$ sharing $e$ as a common edge and $x_0$ as a common vertex. Let $q \in N_M$ be such that $q = q_1 + q_0$, where $q_1 \in C(M)$, $q_1|_T \in P_1(T), q_0|_T \in P_0(T), \forall T \in M$. Thus,

$$0 = \int_M q(\mathrm{div}_c \boldsymbol{v} + r^{-1} v_r) r\, dr\, dz = \frac{1}{2\pi} \int_{\widetilde{M}} \widetilde{q} \nabla \cdot \widetilde{\boldsymbol{v}}\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} = (v_r, v_z) \in \boldsymbol{V}_{0,M},$$

where we used the same notation as in Lemma 4.12. We define

$$\boldsymbol{v}|_{T^\pm} = (\mu \partial_e q)\mathbf{t} = (\mu \partial_e q_1)\mathbf{t},$$
$$\boldsymbol{v}|_T = \boldsymbol{0}, \quad \text{if } T \in \mathcal{T}_h,\ T \neq T^\pm,$$

where $\mathbf{t} = (t_r, t_z)$ is the unit tangential vector to $e$ and $\mu$ is the linear nodal basis function with respect to $\mathcal{T}_{h/2}$, which is one at the midpoint of the edge $e$. With this setting, we obtain the following identity,

$$0 = \int_{\widetilde{M}} \widetilde{q} \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} = \int_{\widetilde{M}} \widetilde{q}_1 \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} + \int_{\widetilde{M}} \widetilde{q}_0 \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x}. \tag{4.33}$$

By the definition of $\boldsymbol{v}$, $\widetilde{\boldsymbol{v}}$ is orthogonal to $\widetilde{\mathbf{n}}$ (normal to $\widetilde{e}$). Therefore, the second term of (4.33) vanishes. Then, we obtain that

$$\begin{aligned}
0 &= \int_{\widetilde{M}} \widetilde{q}_1 \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} = -\int_{\widetilde{M}} \nabla \widetilde{q}_1 \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} \\
&= \int_{\widetilde{M}} (\widetilde{\partial_r q_1} \cos\theta, \widetilde{\partial_r q_1} \sin\theta, \widetilde{\partial_z q_1}) \cdot (\widetilde{\mu} \widetilde{\partial_e q_1} t_r \cos\theta, \widetilde{\mu} \widetilde{\partial_e q_1} t_r \sin\theta, \widetilde{\mu} \widetilde{\partial_e q_1} t_z) \, d\boldsymbol{x} \\
&= \int_{\widetilde{M}} \widetilde{\mu} \widetilde{\partial_e q_1} (\widetilde{\partial_r q_1} t_r + \widetilde{\partial_z q_1} t_z) \, d\boldsymbol{x} = \int_{\widetilde{M}} \widetilde{\mu} |\widetilde{\partial_e q_1}|^2 \, d\boldsymbol{x}.
\end{aligned}$$

The last equation is from the following relation

$$\partial_e q_1 = \partial_r q_1 t_r + \partial_z q_1 t_z. \tag{4.34}$$

Therefore, we have $\partial_e q_1 = 0$ since $\mu \geq 0$. We can also argue that the directional derivative of $q_1$ in any other edge (independent of $e$ in direction) containing $x_0$ is zero. Thus, $q_1$ is constant in $M$, and therefore $q = q_1 + q_0$ is a piecewise constant function in $M$.

It is then enough to show that $q$ is actually a constant in $M$. Similarly to the augmented Hood-Taylor elements, we consider two cases: (I) the common edge $e$ of $T^+$ and $T^-$ is not parallel to the $z$-axis and (II) the common edge $e$ is parallel to the $z$-axis. We shall deal with only the case (I), since the case (II) is similar. For case (I), we define $\boldsymbol{v}$ in the following way:

$$\begin{aligned}
&\boldsymbol{v}|_{T^\pm} = (0, \mu), \\
&\boldsymbol{v}|_T = \mathbf{0}, \quad \text{if } T \in \mathcal{T}_h,\ T \neq T^\pm.
\end{aligned}$$

Clearly, $\boldsymbol{v}$ is a continuous piecewise linear function with respect to $\mathcal{T}_{h/2}$, and zero on $\partial M$. Let $q^\pm$ be the restriction of $q$ to $T^\pm$. We then observe the following relation

$$\begin{aligned}
0 &= \int_{\widetilde{M}} \widetilde{q} \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} = q^+ \int_{\widetilde{T}^+} \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} + q^- \int_{\widetilde{T}^-} \nabla \cdot \widetilde{\boldsymbol{v}} \, d\boldsymbol{x} \\
&= (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\boldsymbol{v}} \cdot \widetilde{\mathbf{n}} \, d\mathbf{s} = (q^+ - q^-) \int_{\widetilde{e}} \widetilde{\mu} n_z \, d\mathbf{s},
\end{aligned}$$

where $\widetilde{\mathbf{n}}$ is the unit normal vector to $\widetilde{e}$ from $\widetilde{T}^+$ to $\widetilde{T}^-$. Since $e$ is not parallel to the $z$-axis, we have $n_z \neq 0$, and therefore $q^+ = q^-$. This process applies to any pair of adjacent triangles in $M$. Hence, we obtain $q$ is a constant function in $M$. Using similar arguments as in Lemma 4.12, $q$ can also be shown to be a constant function in $M$ for case (II). This completes the proof. □

**Remark 4.14.** Using Theorem 4.9, we have shown two new stable mixed methods solving the ASP (the augmented Hood-Taylor elements and the augmented Bercovier-Pironneau elements). Both methods contain piecewise constant functions in the pressure space, and therefore are locally conservative. Consequently, we expect these methods to produce accurate and reliable approximations to the ASP. In addition, according to [8], the augmented Hood-Taylor elements are stable for the 3D Stokes problem when $m \geq 2$. Our result in Lemma 4.12 leads to the new finding that the 3D axisymmetric equation (1.4) can be approximated by using lower-order augmented Hood-Taylor elements ($m \geq 1$).

## 5. Numerical experiments

In this section, we present sample numerical results that confirm our theoretical analysis. We consider the ASP (1.1) in the domain $\Omega = (0,1) \times (0,1)$ with the analytic solution $(\boldsymbol{u}, p) = (u_r, u_z, p) = (r^3 \sin z, 4r^2 \cos z, r^2/2 - 1/4)$. The right hand side data $\boldsymbol{f} = (f_r,\ f_z)$ are calculated based on the given solution. The choice of $p$ is such that it has the zero mean. We note that this section has been inspired by the paper by Boffi et al. [8].

To solve the ASP, we are required to compute the discrete weak formulation. Since a singular function $1/r$ appears in the weak formulation, as given in (2.10), Gaussian quadrature rule (16 Gauss point rule) has been used to alleviate the effect of such function on numerical integration. We implement two tests on mixed methods for the ASP. One test uses the standard lowest-order Hood-Taylor finite element method ($P_2^2 \times P_1$). This method is stable due to [19] but not conservative. The other test uses the lowest-order augmented Hood-Taylor method ($P_2^2 \times P_1 + P_0$). This method is locally conservative for the ASP and its stability is predicated by the theory in this paper. In both cases, let $\mathcal{T}_h$ be a quasi-uniform triangulation with $h$ as the mesh size. Recall the finite element solution $(\boldsymbol{u}_h, p_h)$ of the ASP from (2.16). Let $\widetilde{\Omega} = \Omega \times [-\pi, \pi)$ be the corresponding 3D axisymmetric domain. For a function $v(r,z)$, let $\widetilde{v}(r,\theta,z) = v(r,z)$ be its axisymmetric extension in 3D. Then, $(\widetilde{\boldsymbol{u}}, \widetilde{p}) = (\widetilde{u}_r, 0, \widetilde{u}_z, \widetilde{p})$ is the axisymmetric solution of the 3D Stokes problem with $f_\theta = 0$. Similarly, denote by $(\widetilde{\boldsymbol{u}}_h, \widetilde{p}_h)$ the axisymmetric numerical solution in $\widetilde{\Omega}$ that is the axisymmetric extension of $(\boldsymbol{u}_h, p_h)$.

In Table 1 and Table 2, we list the convergence rates and other quantities of interest provided by the augmented Hood-Taylor method and by the standard Hood-Taylor method, respectively. It is clear from Table 1 that the augmented Hood-Taylor method is stable and has optimal convergence rates ($h^2$ for $\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_h\|_{[H^1(\widetilde{\Omega})]^3} + \|\widetilde{p} - \widetilde{p}_h\|_{L^2(\widetilde{\Omega})}$ and for $\|\nabla \cdot \widetilde{\boldsymbol{u}}_h\|_{L^2(\widetilde{\Omega})}$; $h^3$ for the $L^2$ norm of $\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_h$). This is consistent with our theory in Theorem 4.9 and in Section 4.3.1. Note that Table 2 displays similar results on the convergence rate of the standard Hood-Taylor method. This is also expected since the standard Hood-Taylor methods solving the ASP satisfy the inf-sup condition [19].

**Table 1**
Convergence history of the mixed finite element approximation to the ASP given by the lowest-order augmented Hood-Taylor element ($P_2^2 \times P_1 + P_0$).

| $h$ | $\|\widetilde{u}-\widetilde{u}_h\|_{[L^2]^3}$ | | $\|\widetilde{u}-\widetilde{u}_h\|_{[H^1]^3}$ | | $\|\widetilde{p}-\widetilde{p}_h\|_{L^2}$ | | $\|\nabla\cdot\widetilde{u}_h\|_{L^2}$ | | $\max_{T\in\mathcal{T}_h}\|\int_{\widetilde{T}}\nabla\cdot\widetilde{u}_h dx\|$ |
|---|---|---|---|---|---|---|---|---|---|
| $1/2^2$ | 0.89E-03 | x | 0.60E-01 | x | 0.24E-01 | x | 0.19E-01 | x | 0.36E-07 |
| $1/2^3$ | 0.11E-03 | 3.0 | 0.15E-01 | 2.0 | 0.43E-02 | 2.5 | 0.47E-02 | 2.0 | 0.60E-09 |
| $1/2^4$ | 0.14E-04 | 3.0 | 0.36E-02 | 2.1 | 0.81E-03 | 2.4 | 0.12E-02 | 2.0 | 0.28E-10 |
| $1/2^5$ | 0.17E-05 | 3.0 | 0.90E-03 | 2.0 | 0.17E-03 | 2.3 | 0.29E-03 | 2.1 | 0.56E-10 |
| $1/2^6$ | 0.22E-06 | 3.0 | 0.23E-03 | 2.0 | 0.38E-04 | 2.2 | 0.72E-04 | 2.0 | 0.36E-10 |

**Table 2**
Convergence history of the mixed finite element approximation to the ASP given by the lowest-order Hood-Taylor element ($P_2^2 \times P_1$).

| $h$ | $\|\widetilde{u}-\widetilde{u}_h\|_{[L^2]^3}$ | | $\|\widetilde{u}-\widetilde{u}_h\|_{[H^1]^3}$ | | $\|\widetilde{p}-\widetilde{p}_h\|_{L^2}$ | | $\|\nabla\cdot\widetilde{u}_h\|_{L^2}$ | | $\max_{T\in\mathcal{T}_h}\|\int_{\widetilde{T}}\nabla\cdot\widetilde{u}_h dx\|$ |
|---|---|---|---|---|---|---|---|---|---|
| $1/2^2$ | 0.88E-03 | x | 0.58E-01 | x | 0.17E-01 | x | 0.19E-01 | x | 0.27E-03 |
| $1/2^3$ | 0.11E-03 | 3.0 | 0.14E-01 | 2.1 | 0.34E-02 | 2.3 | 0.47E-02 | 2.0 | 0.18E-04 |
| $1/2^4$ | 0.14E-04 | 3.0 | 0.36E-02 | 2.1 | 0.70E-03 | 2.3 | 0.12E-02 | 2.0 | 0.15E-05 |
| $1/2^5$ | 0.17E-05 | 3.0 | 0.90E-03 | 2.0 | 0.16E-03 | 2.1 | 0.29E-03 | 2.1 | 0.72E-07 |
| $1/2^6$ | 0.22E-06 | 3.0 | 0.23E-03 | 2.0 | 0.36E-04 | 2.2 | 0.72E-04 | 2.0 | 0.45E-08 |

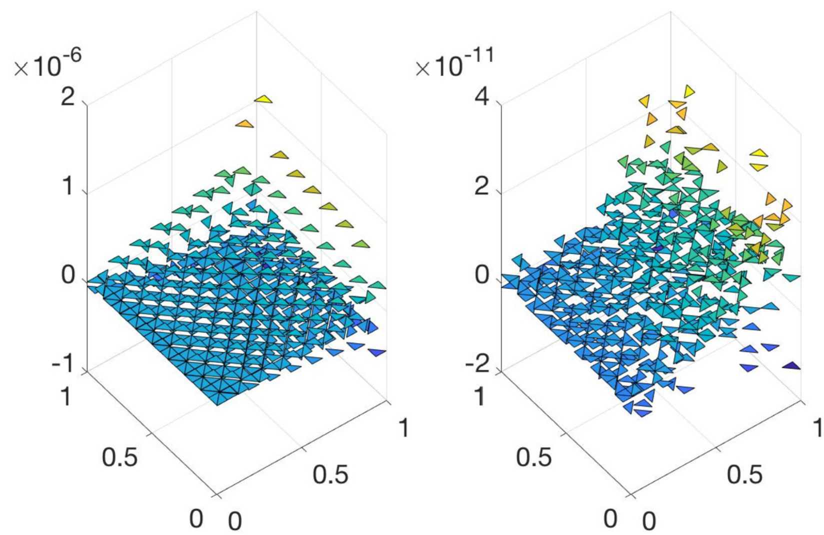


**Fig. 3.** The element-wise value of $\int_T(\mathrm{div}_c \boldsymbol{u}_h + r^{-1}u_{r,h})rdrdz$ from the usual Hood-Taylor method ($P_2^2 \times P_1$) (left) and from the augmented Hood-Taylor method ($P_2^2 \times P_1 + P_0$) (right).

Meanwhile, let us pay more attention to the local conservation property of the augmented Hood-Taylor method, which can be seen by the different test results in the last columns ($\max_{T\in\mathcal{T}_h}|\int_{\widetilde{T}}\nabla\cdot\widetilde{\boldsymbol{u}}_h d\boldsymbol{x}|$) in Table 1 and Table 2. This comparison implies that for the augmented Hood-Taylor method, the local conservation is valid, up to the tolerance used for the solver's stopping criterion (the relative error in a preconditioned MINRES $< 5\times 10^{-10}$). For $\boldsymbol{v}=(v_r, v_z)$ and $\widetilde{\boldsymbol{v}}=(v_r, 0, v_z)$, note that

$$\int_{\widetilde{T}} \nabla\cdot\widetilde{\boldsymbol{v}}d\boldsymbol{x} = 2\pi\int_T (\mathrm{div}_c\boldsymbol{v} + r^{-1}v_r)rdrdz.$$

We also plot the values of the local integrals $\int_T(\mathrm{div}_c\boldsymbol{u}_h + r^{-1}u_{r,h})rdrdz$ (the axisymmetric divergence of the numerical velocity) from different methods in Fig. 3. On the triangulation of mesh size $h = 1/2^4$, these plots clearly demonstrate the local conservative property in the augmented Hood-Taylor method with much smaller values.

In addition, according to Lemma 4.12, the assumption that each triangle has at least one interior node is important in the proof of the stability of the mixed method. To conclude this section, we demonstrate in Fig. 4 the effect of this assumption on the stability of the pressure approximation by the augmented Hood-Taylor elements.

## 6. Conclusions

In this paper, we developed technical tools to carry out the macroelement analysis on mixed finite element approximations of the ASP. In particular, we provided local macroelement conditions (Assumption 4.2) that are sufficient to verify the well-posedness (the inf-sup condition) of the mixed method. Thus, we generalized the macroelement technique for the usual Stoke equations [26] to the ASP. The main difficulty in analysis is due to the singular and vanishing weights in the bilinear form and in the function spaces. Therefore, new estimates in weighted spaces had to be established to obtain the desired result. This led to a different macroelement condition than the one for the usual Stokes problem [26]. Meanwhile, the use of overlapping macroelements (Definition 4.1) is supported by estimates regarding multiple projection operators in Lemma 4.7

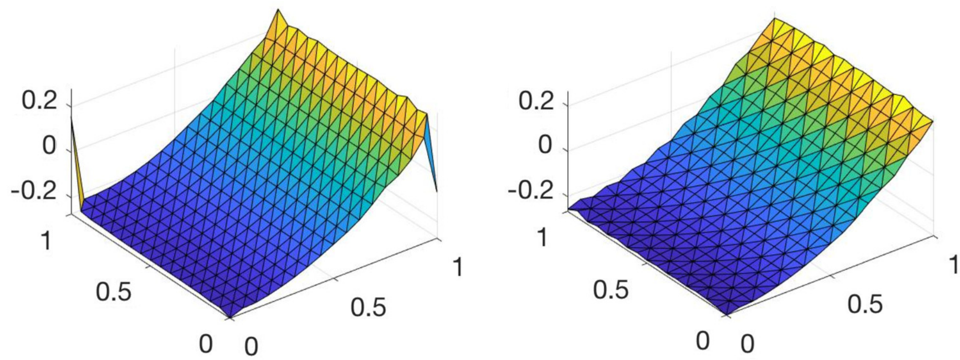


**Fig. 4.** Discrete pressures obtained by applying the augmented Hood-Taylor method ($P_2^2 \times P_1 + P_0$) using triangulations with elements having no interior node (left) and triangulations with elements having interior nodes (right).

and Lemma 4.8. This is also different from [26], where the estimates were for one projection operator and were assumed on non-overlapping macroelements. The immediate consequence of our result is a local condition that can be used to develop stable mixed finite element methods for the ASP. As an application of the proposed macroelement condition, we provided new stable mixed finite element methods for the ASP, which are locally conservative. Numerical test results verified the theoretical prediction. We mention that some other axisymmetric equations, such as the axisymmetric linear elasticity equations and the axisymmetric Maxwell equations, are defined in similar weighted spaces. Therefore, we also expect the estimates in this paper will motivate new numerical techniques for these axisymmetric problems.

## Link to the Reproducible Capsule

https://codeocean.com/capsule/0180711/

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