

# Uniform Shift Estimates for Transmission Problems and Optimal Rates of Convergence for the Parametric Finite Element Method

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**Abstract.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , be a bounded domain with piecewise smooth boundary  $\partial\Omega$  and let  $U$  be an open subset of a Banach space  $Y$ . Motivated by questions in “Uncertainty Quantification,” we consider a parametric family  $P = (P_y)_{y \in U}$  of uniformly strongly elliptic, second order partial differential operators  $P_y$  on  $\Omega$ . We allow jump discontinuities in the coefficients. We establish a regularity result for the solution  $u : \Omega \times U \rightarrow \mathbb{R}$  of the parametric, elliptic boundary value/transmission problem  $P_y u_y = f_y$ ,  $y \in U$ , with mixed Dirichlet-Neumann boundary conditions in the case when the boundary and the interface are smooth and in the general case for  $d = 2$ . Our regularity and well-posedness results are formulated in a scale of broken weighted Sobolev spaces  $\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega)$  of Babuška-Kondrat’ev type in  $\Omega$ , possibly augmented by some locally constant functions. This implies that the parametric, elliptic PDEs  $(P_y)_{y \in U}$  admit a shift theorem that is uniform in the parameter  $y \in U$ . In turn, this then leads to  $h^m$ -quasi-optimal rates of convergence (*i. e.*, algebraic orders of convergence) for the Galerkin approximations of the solution  $u$ , where the approximation spaces are defined using the “polynomial chaos expansion” of  $u$  with respect to a suitable family of tensorized Lagrange polynomials, following the method developed by Cohen, Devore, and Schwab (2010).

## 1 Introduction

Recently, questions related to differential equations with random coefficients have received a lot of attention due to the practical applications of these problems

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[2,7,9,16,17]. Our paper is motivated by the approach in [6,13], where families of differential operators on polyhedral domains indexed by  $y \in U$ , were studied. As in those papers,  $U$  is an open subset of a Banach space  $Y$ , which allows us to study the analyticity of the solution in terms of  $y \in U$ .

We here study the properties of solutions to a family of strongly elliptic, mixed boundary value/transmission problems

$$P_y u_y(x) = Pu(x, y) = f_y(x) = f(x, y), \quad x \in \Omega, \quad y \in U \quad (1)$$

on a domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ . The domain  $\Omega$  is assumed to be piecewise smooth and bounded. Thus, for each  $y \in U$ , we are given a second order, uniformly strongly positive, parametric partial differential operator  $P_y$  on  $\Omega$  whose coefficients are functions of  $(x, y) \in \Omega \times U$  and are allowed to have jump discontinuities across a fixed interface  $\Gamma$ . More precisely, we assume that  $\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k$ , where  $\Omega_k$  are disjoint domains with piecewise smooth boundaries and  $\Gamma := (\cup_{k=1}^K \partial\Omega_k) \setminus \partial\Omega$ .

Under suitable regularity assumptions on the coefficients of  $P$  and on the source term  $f : \Omega \times U \rightarrow \mathbb{R}$ , we establish in Section 5 a regularity and well-posedness result for the solution  $u : \Omega \times U \rightarrow \mathbb{R}$  of the parametric, elliptic boundary value/transmission problem (1) with mixed Dirichlet-Neumann boundary conditions. Our regularity result is formulated in a scale of broken weighted Sobolev spaces  $\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) = \oplus_{k=1}^K \mathcal{K}_{a+1}^{m+1}(\Omega_k)$  of Babuška-Kondrat'ev type in  $\Omega$ , for which we prove that our elliptic PDEs  $(P_y)_{y \in U}$  admit a shift theorem that is uniform in the parameter  $y \in U$ . We deal completely in this paper with the cases when the boundary  $\partial\Omega$  and the interface  $\Gamma$  are smooth and disjoint. We also indicate how to proceed in the general case for  $d = 2$ . Our results generalize the results of [13] by allowing jump discontinuities in the coefficients and by allowing adjacent edges to be endowed with Neumann-Neumann boundary conditions. We will be therefore brief in our presentation, referring to [13], as well as [6,7] for more details.

The main contribution of this paper is to study the regularity of the solution of a (non-parametric) transmission/boundary value problem with rather weak smoothness assumptions on the coefficients. As far as we know, this paper is the only place where a complete proof for the regularity of transmission problems is given, even in the case of smooth coefficients. The results are general enough so that one can use the approach in [6,13] to obtain regularity results for families and then to obtain optimal rates of convergence for the Galerkin method. An abstract version of this method is explained in [3]. These issues will be discussed in more detail in a forthcoming paper.

The paper is organized as follows. In Section 2 we formulate our parametric partial differential boundary value/transmission problem and introduce some of our main assumptions. We also discuss the needed notions of positivity for families of operators and derive some simple consequences. In Section 3, we review the ‘‘broken’’ version of usual Sobolev spaces, and then formulate and prove the main results, Theorem 1, which is a regularity and well-posedness result for non-parametric solutions in smooth case. In Section 4, we recapitulate

regularity and well-posedness results for the non-parametric, elliptic problem from [4,5,10], the main result being Theorem 2. This theorem is then generalized to families in Section 5, thus yielding our main regularity and well-posedness result for parametric families of uniformly strongly boundary value/transmission problems, namely Theorem 3. As mentioned above, this result is formulated in broken weighted Sobolev spaces (the so called “Babuška-Kondrat’ev” spaces). As in [6,7,13] these results lead to  $h^m$ -quasi-optimal rates of convergence for a suitable Galerkin method for the approximation of our parametric solution  $u$ .

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## 2 Ellipticity, Positivity, Solvability for Parametric Families

We now formulate our parametric partial differential boundary value/transmission problem and introduce some of our main assumptions.

### 2.1 Notation and Assumptions

By  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , we shall denote a connected, bounded piecewise smooth domain, which we assume is decomposed into finitely many subdomains  $\Omega_k$  with piecewise smooth boundary,  $\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k$ . We obtain results on the spatial regularity of PDEs whose data depend on a parameter vector  $y \in U \subset Y$ , where  $U$  is an open subset of a Banach space  $Y$ . By  $a_{pq}^{ij}, b_{pq}^i, c_{pq} : \Omega \times U \rightarrow \mathbb{R}$ ,  $1 \leq i, j \leq d$ , we shall denote bounded, measurable functions satisfying smoothness and other assumptions to be made precise later. We denote by  $A = (a_{pq}^{ij}, b_{pq}^i, c_{pq})$ . Let us denote by  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $i = 1, \dots, d$ . We shall then denote by  $P^A = [P_{pq}^A]$  a  $\mu \times \mu$  matrix of parametric differential operators in divergence form

$$P_{pq}^A u(x, y) := \left( - \sum_{i,j=1}^d \partial_i (a_{pq}^{ij}(x, y) \partial_j) + \sum_{i=1}^d b_{pq}^i(x, y) \partial_i + c_{pq}(x, y) \right) u(x, y), \quad (2)$$

where  $x \in \Omega$  and  $y \in U$ . Note that the derivatives act only in the  $x$ -direction and  $y$  is just a parameter. The matrix case is needed in order to handle the case of systems, such as that of (anisotropic) linear elasticity.

A matrix  $P = [P_{pq}]_{p,q=1}^\mu$  of differential operators acts on vector-valued functions  $u = (u_q)_{q=1}^\mu$  in the usual way  $(Pu)_p = \sum_{q=1}^\mu P_{pq} u_q$ , for  $u = (u_q) \in \mathcal{C}^\infty(\Omega \times U)^\mu$ . We recall that  $H^{-1}(\Omega)$  is defined as the dual of  $H_0^1(\Omega) := \{u \in H^1(\Omega), u|_{\partial\Omega} = 0\}$  with pivot  $L^2(\Omega)$ . Occasionally, we shall need to specialize a family  $P$  for a particular value of  $y$ , in which case we shall write  $P_y : \mathcal{C}^\infty(\Omega)^\mu \rightarrow H^{-1}(\Omega)^\mu$  for the induced operator. We emphasize that we allow  $P$  to have non-smooth coefficients, so that  $Pu$  may be non-smooth in general.

## 2.2 Boundary and Interface Conditions

We impose mixed Dirichlet and Neumann boundary conditions. To this end, we assume there is given a closed set  $\partial_D\Omega \subset \partial\Omega$ , which is a union of polygonal subsets of the boundary and we let  $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ . The set  $\partial_D\Omega$  will be referred to as “Dirichlet boundary” and  $\partial_N\Omega$  as “Neumann boundary,” according to the type of boundary conditions that we associate to these parts of the boundary. The case of cracks is also allowed, provided that one treats different sides of the crack as *different* parts of the boundary, as in [10], for instance, but we choose not to treat this case explicitly in this paper. We then define the *conormal derivatives*

$$(\nabla_\nu^A u)_p(x, y) = \sum_{q=1}^{\mu} \sum_{i,j=1}^d \nu_i a_{pq}^{ij}(x, y) \partial_j u_q(x, y), \quad x \in \partial_N\Omega, y \in U, \quad (3)$$

where  $\nu = (\nu_i)$  is the outward unit normal vector at  $x \in \partial_N\Omega$ . The conormal derivatives  $\nabla_\nu^A u^\pm$  at the interface  $\Gamma$  are defined similarly, using an arbitrary but fixed labeling of the two sides of the interface into a positive and a negative part.

We shall also need the spaces  $H_D^1(\Omega)$  and  $H_D^{-1}(\Omega)$  for vector-valued functions:  $H_D^1(\Omega) := \{u \in H^1(\Omega)^\mu, u = 0 \text{ on } \partial_D\Omega\}$  and  $H_D^{-1}(\Omega)$ , defined to be the dual of  $H_D^1(\Omega)$  with pivot space  $L^2(\Omega)$ . Note that we assume here as in [13] that we have *the same type of boundary conditions for all components  $u_q$  of the solution vector  $u$ .*

Recall that our domain  $\Omega$  is decomposed into  $K$  subdomains of the same type (with piecewise smooth boundary),  $\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k$ . We then denote by  $\Gamma := (\cup_{k=1}^K \partial\Omega_j) \setminus \partial\Omega$  the interface of our problem. We also fix arbitrarily the sides of the interface, and we thus denote by  $u^+$ , respectively  $u^-$  the non-tangential limits of  $u$  at the two sides of the interface. We define similarly the conormal derivatives  $\nabla_\nu^A u^+$  and  $\nabla_\nu^A u^-$  at the interface, but using the two sided limits of the coefficients  $a_{pq}^{ij}$  at  $\Gamma$ . We consider the parametric family of boundary value/interface problems

$$\begin{cases} P^A u(x, y) = f(x, y) & x \in \Omega, \\ u(x, y) = 0 & x \in \partial_D\Omega, \\ \nabla_\nu^A u(x, y) = g(x, y) & x \in \partial_N\Omega \\ u^+(x, y) - u^-(x, y) = 0 & x \in \Gamma \\ \nabla^A u^+(x, y) - \nabla^A u^-(x, y) = h(x, y) & x \in \Gamma \end{cases} \quad (4)$$

where  $P^A$  is as in Equation (2),  $\nabla_\nu^A$  is as in Equation (3), and  $y \in U$ . We stress that for us the dependence of  $P^A$  on its coefficients, that is on  $A$ , is important, which justifies our notation.

## 2.3 Ellipticity and Positivity for Differential Operators

In this subsection we recall the definition of the positivity property for parametric families of differential operators. Let us therefore consider, for any  $y \in U$ , the parametric bilinear form  $B(y; \cdot, \cdot)$  defined by

$$B(y; v, w) := \int_{x \in \Omega} \sum_{p, q=1}^{\mu} \left( \sum_{i, j=1}^d a_{pq}^{ij}(x, y) \partial_i v_p(x, y) \partial_j w_q(x, y) \right. \\ \left. \sum_{i=1}^d b_{pq}^i(x, y) \partial_i v_p(x, y) w_q(x, y) + c_{pq}(x, y) v_p(x, y) w_q(x, y) \right) dx, \quad y \in U. \quad (5)$$

**Definition 1.** The family  $(P_y)_{y \in U}$  is called uniformly strictly positive definite on  $H_0^1(\Omega)^\mu \subset V \subset H^1(\Omega)^\mu$  if the coefficients  $a_{pq}^{ij}$  are symmetric in  $i, j$  and in  $p, q$  (that is,  $a_{pq}^{ij} = a_{pq}^{ji} = a_{qp}^{ij}$ , for all  $i, j, p$ , and  $q$ ), and if there exist  $0 < r < R < \infty$  such that for all  $y \in U$ , and  $v, w \in V$ , we have

$$|B(y; v, w)| \leq R \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \quad \text{and} \quad r \|v\|_{H^1(\Omega)}^2 \leq B(y; v, v).$$

If  $U$  is reduced to a single point, that is, if we deal with the case of a single operator instead of a family, then we say that  $P$  is strictly positive definite. Throughout this paper, we shall assume that  $(P_y)_{y \in U}$  is uniformly strictly positive definite. Positivity is closely related to ellipticity.

**Definition 2.** The family  $(P_y)_{y \in U}$  is called uniformly strongly elliptic if the coefficients  $a_{pq}^{ij}$  are symmetric in  $i, j$  and in  $p, q$  and if there exist  $0 < r_e < R_e < \infty$  such that for all  $x \in D$ ,  $y \in U$ ,  $\xi \in \mathbb{R}^d$ , and  $\eta \in \mathbb{R}^\mu$

$$r_e |\xi|^2 |\eta|^2 \leq \sum_{p, q=1}^{\mu} \sum_{i, j=1}^d a_{pq}^{ij}(x, y) \xi_i \xi_j \eta_p \eta_q \leq R_e |\xi|^2 |\eta|^2. \quad (6)$$

In case one is interested only in scalar equations (not in systems), then for  $V = H_D^1(\Omega)$ , the assumption that our family  $P_y$  is uniformly positive definite can be replaced with the (slightly weaker) assumption that the family  $P_y$  is uniformly strongly elliptic, that  $\sum_{i=1}^d \partial_i b^i = 0$  in  $\Omega$ ,  $\sum_{i=1}^d \nu_i b^i = 0$  on  $\partial_N \Omega$ ,  $c \geq 0$ , and  $\partial_D \Omega \neq \emptyset$  (in which case it also follows that  $\partial_D \Omega$  has a non-empty measure). In general, a uniformly strictly positive family  $P$  will also be uniformly strongly elliptic.

## 2.4 Consequences of Positivity

The usual Lax-Milgram lemma gives the following result as in [6]. Recall the constant  $r$  from Definition 1.

**Proposition 1.** Assume that  $f_y := f(\cdot, y) \in H_D^{-1}(\Omega)$ , for any  $y \in U$ . Also, assume that the family  $P$  is uniformly strictly positive definite. Then our family of boundary value problems  $P_y u_y = f_y$ ,  $u_y \in H_D^1(\Omega)$ , i. e., Equation (4), admits a unique solution  $u_y = P_y^{-1} f_y$ . Moreover,  $\|P_y^{-1}\|_{\mathcal{L}(H_D^{-1}; H_D^1)} \leq r^{-1}$ , for all  $y \in U$ .

The parametric solution  $u_y \in H_D^1(\Omega)$  of Proposition 1 is then obtained from the usual weak formulation: given  $y \in U$ , find  $u_y \in V := H_D^1(\Omega)$  such that

$$B(y; u_y, w) = (f_y, w) + \int_{\partial_N \Omega} g_y w dS + \int_{\Gamma} h_y w dS, \quad \forall w \in V, \quad (7)$$

where  $(f_y, w)$  denotes the  $L^2(\Omega)$  inner product and  $dS$  is the surface measure on  $\partial \Omega$  or on  $\Gamma$ . Also,  $f_y(x) = f(x, y)$ , and similarly for  $u_y$ ,  $g_y$ , and  $h_y$ .

### 3 Broken Sobolev Spaces and Higher Regularity of Non-parametric Solutions in the Smooth Case

One of our main goals is to obtain regularity of the solution  $u$  both in the space variable  $x$  and in the parameter  $y$ . It is convenient to split this problem into two parts: regularity in  $x$  and regularity in  $y$ . We first address regularity in  $x$  in the case when the boundary  $\partial\Omega$  and the interface  $\Gamma$  are smooth and disjoint. We also assume that each connected component of the boundary is given a single type of boundary conditions: either Dirichlet or Neumann. This leads to Theorem 1, which states the regularity and well-posedness of Problem (4) in this smooth case ( $\partial\Omega$  and  $\Gamma$  smooth and disjoint). This is the main result of this paper, and, as far as we know, no complete proof was given before. We also consider coefficients with lower regularity than it is usually assumed, which is needed to treat the truly parametric case. (We are planning to address this question in a future paper.)

*We assume throughout this and the following section that we are dealing with a single, non-parametric equation (not with a family), that is, that  $U$  is reduced to a single point in this subsection. We also assume that  $\mu = 1$ , to simplify the notation.*

We shall need the “broken” version of the usual Sobolev spaces to deal with our interface problem. Recall the subdomains  $\Omega_k \subset \Omega$ ,  $1 \leq k \leq K$ , we define:

$$\begin{aligned} \hat{H}^m(\Omega) &:= \{v : \Omega \rightarrow \mathbb{R}, v \in H^m(\Omega_k), \forall 1 \leq k \leq K\} \\ \hat{W}^{m,\infty}(\Omega) &:= \{v : \Omega \rightarrow \mathbb{R}, \partial^\alpha v \in L^\infty(\Omega_k), \forall 1 \leq k \leq K, |\alpha| \leq m\}. \end{aligned} \quad (8)$$

For further reference we note that the definitions of these spaces imply that the multiplication and differentiation maps  $\hat{W}^{m,\infty}(\Omega) \times \hat{H}^m(\Omega) \rightarrow \hat{H}^m(\Omega)$  and  $\partial_i : \hat{H}^m(\Omega) \rightarrow \hat{H}^{m-1}(\Omega)$  are continuous.

One of the difficulties of dealing with interface problems is the more complicated structure of the domains and ranges of our operators. When  $m = 0$ , we define  $\mathcal{D}_m = \mathcal{D}_0 = H_D^1(\Omega) = V$  and  $\mathcal{R}_m = \mathcal{R}_0 = H_D^{-1}(\Omega) = V^*$ . Then we define  $\tilde{P}^A$  in a weak sense using the bilinear form  $B$  introduced in Equation (5) (see the discussion around Equation (2.12) in [10] for more details or the discussion around Equation (20) in [11]). Assume now that  $m \geq 1$ . We then define

$$\begin{aligned} \mathcal{D}_m &:= \hat{H}^{m+1}(\Omega) \cap \{u = 0 \text{ on } \partial_D \Omega\} \cap \{u^+ - u^- = 0 \text{ on } \Gamma\} \text{ and} \\ \mathcal{R}_m &:= \hat{H}^{m-1}(\Omega) \oplus H^{m-1/2}(\partial_N \Omega) \oplus H^{m-1/2}(\Gamma). \end{aligned}$$

In particular,  $\mathcal{D}_m = \hat{H}^{m+1}(\Omega) \cap H_D^1(\Omega)$ . Let  $A = (a^{ij}, b^i, c) \in \hat{W}^{m,\infty}(\Omega)^{d^2+d+1}$  and  $P^A u = \sum_{i,j=1}^2 \partial_i(a^{ij} \partial_j u) + \sum_{i=1}^2 b^i \partial_i u + cu$ , as before. Then the family of partial differential operators  $\tilde{P}_m^A : \mathcal{D}_m \rightarrow \mathcal{R}_m$ ,

$$\tilde{P}_m^A u = (P^A u, \nabla_\nu^A u|_{\partial_N \Omega}, (\nabla_\nu^A u^+ - \nabla_\nu^A u^-)|_\Gamma) \quad (9)$$

is well defined. Note that the domain  $\mathcal{D}_m$  and codomain  $\mathcal{R}_m$  are independent of  $y \in U$ , which justifies why we do not consider homogeneous Neumann boundary

conditions. We are now ready to state and prove our main theorem. Let us denote  $\|u\|_{\hat{H}^m(\Omega)} := \left( \sum_{k=1}^K \|u\|_{H^m(\Omega_k)}^2 \right)^{1/2}$  and  $\|v\|_{\hat{W}^{m,\infty}(\Omega)} := \sum_{k=1}^K \|v\|_{W^{m,\infty}(\Omega_k)}$  the resulting natural norms on the spaces introduced in Equation (8).

**Theorem 1.** *Let us assume that  $\Omega \subset \mathbb{R}^d$  is smooth and bounded, that the interface  $\Gamma$  is smooth and does not intersect the boundary, and that to each component of the boundary it is associated a single type of boundary conditions (either Dirichlet or Neumann). Assume that  $A = (a^{ij}, b^i, c) \in \hat{W}^{m,\infty}(\Omega)^{d^2+d+1}$  and that  $P^A$  is strictly positive definite on  $H_D^1(\Omega)$ , then  $\tilde{P}_m^A : \mathcal{D}_m \rightarrow \mathcal{R}_m$  is invertible.*

*Moreover, let  $\|P^{-1}\|$  denote norm of the inverse of the map  $P : H_D^1(\Omega) \rightarrow H_D^1(\Omega)^* =: H_D^{-1}(\Omega)$ . Then there exists a constant  $\tilde{C}_1 > 0$  such that the solution  $u$  of (4) satisfies*

$$\|u\|_{\hat{H}^{m+1}(\Omega)} + \|u\|_{H^1(\Omega)} \leq \tilde{C}_1 \left( \|f\|_{\hat{H}^{m-1}(\Omega)} \|g\|_{H^{m-\frac{1}{2}}(\partial_N \Omega)} + \|h\|_{H^{m-\frac{1}{2}}(\Gamma)} \right), \quad (10)$$

with the constant  $\tilde{C}_1 = \tilde{C}_1(m, \|P^{-1}\|, \|A\|_{\hat{W}^{m,\infty}})$ .

*Proof.* In the case of the pure Dirichlet boundary conditions for an equation and without the explicit bounds in Equation (10), this lemma is a classical result, which is proved using divided differences and the so called ‘‘Nirenberg’s trick’’ (see [8,12]). Since we consider transmission problems and want the more explicit bounds in the above Equation (10), let us now indicate the main steps to treat the interface regularity following the classical proof and [13]. The boundary conditions (*i. e.*, regularity at the boundary) were dealt with in [13]. In all the calculations below, all the constants  $C$  in this proof will be generic constants that will depend only on the variables on which  $\tilde{C}_1$  depends (*i. e.*, on the order  $m$ , the norms  $\|P^{-1}\|$  and  $\|A\|_{\hat{W}^{m,\infty}(\Omega)}$ ). We split the proof into several steps.

*Step 1.* We first use Proposition 1 to conclude that  $P : H_D^1(\Omega) \rightarrow H_D^1(\Omega)^*$  is indeed invertible. This provides the needed estimate for  $m = 0$  (in which case, we recall, our problem (4) has to be interpreted in a weak sense).

*Step 2.* For  $m > 0$  we can assume  $g = 0$  and  $h = 0$  by using the extension theorem as in [13].

*Step 3.* We also notice that, in view of the invertibility of  $P$  for  $m = 0$  and since  $u \in H_D^1(\Omega)$ , it suffices to prove

$$\|u\|_{\hat{H}^{m+1}(\Omega)} \leq C \left( \sum_k \|f\|_{H^{m-1}(\Omega_k)} + \|u\|_{\hat{H}^m(\Omega)} \right). \quad (11)$$

Indeed, the desired inequality (10) will follow from Equation (11) by induction on  $m$ . Since Equation (11) holds for  $P$  if, and only if, it holds for  $\lambda + P$ , in order to prove Equation (11), it is also enough to assume that  $\lambda + P$  is strictly positive for some  $\lambda \in \mathbb{R}$ . In particular, Equation (11) will continue to hold—with possibly different constants—if we change the lower order terms of  $P$ .

*Step 4.* Let us assume that  $\Omega = \mathbb{R}^d$  with the interface given by  $\Gamma = \{x_d = 0\}$ . Let  $\Omega_1 = \mathbb{R}_+^d$  and  $\Omega_2 = \mathbb{R}_-^d$  be the two halves into which  $\mathbb{R}^d$  is divided (so  $K = 2$ ). Then we prove Equation (11) for these particular domains and for

$g = 0$  and  $h = 0$  by induction on  $m$ . As we have noticed, the Equation (11) is true for  $m = 0$ , since the stronger relation (10) is true in this case. Thus, we shall assume that Equation (11) has been proved for  $m$  and for smaller values and we will prove it for  $m + 1$ . That is, we want to prove

$$\|u\|_{\hat{H}^{m+2}(\mathbb{R}^d)} \leq C(\|f\|_{\hat{H}^m(\mathbb{R}^d)} + \|u\|_{\hat{H}^{m+1}(\mathbb{R}^d)}). \quad (12)$$

To this end, let us first write

$$\|u\|_{\hat{H}^{m+2}(\mathbb{R}^d)} \leq \sum_{j=1}^d \|\partial_j u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}. \quad (13)$$

We then use our estimate (11) for  $m$  (using the induction hypothesis) applied to the function  $\partial_j u$  for  $j < d$ . This gives

$$\begin{aligned} \|\partial_j u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} &\leq \|P\partial_j u\|_{\hat{H}^{m-1}(\mathbb{R}^d)} \leq \|\partial_j f\|_{\hat{H}^{m-1}(\mathbb{R}^d)} + \|[P, \partial_j]u\|_{\hat{H}^{m-1}(\mathbb{R}^d)} \\ &\leq \|f\|_{\hat{H}^m(\mathbb{R}^d)} + C\|u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} \end{aligned} \quad (14)$$

since the commutator  $[P, \partial_j] = P\partial_j - \partial_j P$  is an operator of order  $\leq 2$  whose coefficients can be bounded in terms of  $\|A\|_{\dot{W}^{m,\infty}(\Omega)}$ . We now only need to estimate  $\|\partial_d u\|_{\hat{H}^{m+1}}$ , we do that on each half subspace.

$$\begin{aligned} \|\partial_d u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} &\leq \sum_{j=1}^d \|\partial_j \partial_d u\|_{\hat{H}^m(\mathbb{R}^d)} + \|\partial_d u\|_{L^2(\mathbb{R}^d)} \\ &\leq \sum_{j=1}^{d-1} \|\partial_j u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} + \|\partial_d^2 u\|_{\hat{H}^m(\mathbb{R}^d)} + \|u\|_{\hat{H}^1(\mathbb{R}^d)}. \end{aligned} \quad (15)$$

The right hand side of the above equation contains only terms that have already been estimated in the desired way, except for  $\|\partial_d^2 u\|_{\hat{H}^m}$ . Since  $m \geq 0$ , we can use the relation  $Pu = f$  to estimate this term as follows. Let us write  $Pu = \sum \partial_i (a^{ij} \partial_j u) + \sum b^i \partial_i u + cu$ . This gives  $a^{dd} \partial_d^2 u = f - \sum_{(i,j) \neq (d,d)} a^{ij} \partial_i \partial_j u + Qu$ , where  $Q$  is a first order differential operator. Next we notice that  $a^{dd}$  is uniformly bounded from below by the uniform strong positivity property (which implies uniform strong ellipticity):  $(a^{dd})^{-1} \leq r^{-1}$ . Note that by Proposition 1, we have  $\|P^{-1}\| \leq r$ , and hence  $r^{-1}$  is an admissible constant. This gives  $\partial_d^2 u = (a^{dd})^{-1} f - \sum_{j=1}^{d-1} B^j \partial_j u + Q_1 u$  where  $B^j$  and  $Q_1$  are first order differential operators with coefficients bounded by admissible constants, which then gives

$$\begin{aligned} \|\partial_d^2 u\|_{\hat{H}^m(\mathbb{R}^d)} &\leq C(\|f\|_{\hat{H}^m(\mathbb{R}^d)} + \sum_{j=1}^{d-1} \|\partial_j u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} + \|u\|_{\hat{H}^{m+1}(\mathbb{R}^d)}) \\ &\leq C(\|f\|_{\hat{H}^m(\mathbb{R}^d)} + \|u\|_{\hat{H}^{m+1}(\mathbb{R}^d)}) \end{aligned} \quad (16)$$

by Equation (14). Equation (15) and (16) then give

$$\|\partial_d u\|_{\hat{H}^{m+1}(\mathbb{R}^d)} \leq C(\|f\|_{\hat{H}^m(\mathbb{R}^d)} + \|u\|_{\hat{H}^{m+1}(\mathbb{R}^d)}). \quad (17)$$

Combining Equations (17) and (14) with Equation (13) gives then the desired Equation (12) for  $m$  replaced with  $m + 1$ .



*Step 5.* We finally reduce to the case of a half-space or a full space using a partition of unity as in the classical case, as follows. We choose a smooth partition of unity  $(\phi_j)$  on  $\Omega$  consisting of functions with small supports. The supports should be small enough so that if the support of  $\phi_j$  intersects the boundary of  $\Omega$  or the interface  $\Gamma$ , then the boundary or the interface can be straightened in a small neighborhood of the support of  $\phi_j$ . We arrange that the resulting operators are positive and we complete the proof as in [13].

See also [14,15].

## 4 Weighted Sobolev Spaces and Higher Regularity of Non-parametric Solutions

We now assume  $d = 2$ , so  $\Omega$  is a plane domain. We allow however  $\Omega$  to be *piecewise* smooth. We also consider coefficients with lower regularity than the ones considered in [10]. This leads to Theorem 2, which will be then generalized to families in a forthcoming paper, which will contain also full details for the remaining results. *We continue to assume that we are dealing with a single, non-parametric equation and that  $\mu = 1$ .*

To formulate further assumptions on our problem and to state our results, we shall need weighted Sobolev spaces, both of  $L^2$  and of  $L^\infty$  type. Let  $\mathcal{V}$  be the set of singular points, where  $Q \in \mathcal{V}$  if one of the following is satisfied: (1) it is a vertex, (2) it is a point where the boundary condition changes type (from Dirichlet to Neumann), (3) it is a point where the interface meets the boundary, or (4) it is a non-smooth point on the interface of the subdomains  $\Omega_k$ . Let us denote from now on by  $\rho : \mathbb{R}^2 \rightarrow [0, 1]$  a continuous function that is smooth outside the set  $\mathcal{V}$  and is such that  $\rho(x)$  is equal to the distance from  $x \in \mathbb{R}^2$  to  $\mathcal{V}$  when  $x$  is close to the singular set  $\mathcal{V}$ . The function  $\rho$  will be called the *smoothed distance to the set of singular points*. We can also assume  $\|\nabla \rho\| \leq 1$ , which will be convenient in later estimates, since it will reduce the number of constants (or parameters) in our estimates. We first define the *Babuška-Kondrat'ev spaces*

$$\mathcal{K}_a^m(\Omega) := \{v : \Omega \rightarrow \mathbb{R}, \rho^{|\alpha|-a} \partial^\alpha v \in L^2(\Omega), \forall |\alpha| \leq m\} \quad (18)$$

$$\mathcal{W}^{m,\infty}(\Omega) := \{v : \Omega \rightarrow \mathbb{R}, \rho^{|\alpha|} \partial^\alpha v \in L^\infty(\Omega), \forall |\alpha| \leq m\}. \quad (19)$$

We shall denote by  $\|\cdot\|_{\mathcal{K}_a^m(\Omega)}$  and  $\|\cdot\|_{\mathcal{W}^{m,\infty}(\Omega)}$  the resulting natural norms on these spaces. We shall need also the “broken” version of these Babuška-Kondrat'ev spaces for our interface problem. Recall the subdomains  $\Omega_k \subset \Omega$ ,  $1 \leq k \leq K$ . In analogy with the smooth case, we then define:  $\hat{\mathcal{K}}_a^m(\Omega) := \{v : \Omega \rightarrow \mathbb{R}, v \in \mathcal{K}_a^m(\Omega_k), \forall 1 \leq k \leq K\}$ , and  $\hat{\mathcal{W}}^{m,\infty}(\Omega) := \{v : \Omega \rightarrow \mathbb{R}, v \in \mathcal{W}^{m,\infty}(\Omega_k), \forall 1 \leq k \leq K\}$ . If  $\mathcal{V}$  is empty (that is, if the domain  $\Omega$  is smooth and the interface is also smooth and does not touch the boundary), then we set  $\rho \equiv 1$  and our spaces reduce to the broken Sobolev spaces  $\hat{H}^m(\Omega)$  and  $\hat{W}^{m,\infty}$  introduced in the previous section, Equation (8). As in the smooth case, the multiplication and differentiation maps  $\hat{\mathcal{W}}^{m,\infty}(\Omega) \times \hat{\mathcal{K}}_a^m(\Omega) \rightarrow \hat{\mathcal{K}}_a^m(\Omega)$  and  $\partial_i : \hat{\mathcal{K}}_a^m(\Omega) \rightarrow \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega)$  are continuous. Let  $S \subset \partial\Omega_k$ . Also as in the smooth

case, we define the spaces  $\mathcal{K}_{a+1/2}^{m+1/2}(S)$  as the restrictions to  $S$  of the functions  $u \in \mathcal{K}_{a+1}^{m+1}(\Omega_k)$ . These spaces have intrinsic descriptions [1,11] similar to the usual Babuška-Kondrat'ev spaces. Note that no “hat” is needed for the boundary version of the spaces  $\hat{\mathcal{K}}$ . Also  $\mathcal{K}_{a+1/2}^{m+1/2}(S_1 \cup S_2) = \mathcal{K}_{a+1/2}^{m+1/2}(S_1) \oplus \mathcal{K}_{a+1/2}^{m+1/2}(S_2)$ , if  $S_1$  and  $S_2$  are disjoint.

We need to consider the subset  $\mathcal{V}_s$  of  $\mathcal{V}$  consisting of Neumann-Neumann corners (*i. e.*, corners where two edges endowed with Neumann boundary conditions meet) and non-smooth points of the interface, which can be described as  $\mathcal{V}_s := \mathcal{V} \setminus \{Q \in \mathcal{V}, Q \in \overline{\partial_D \Omega}\}$ . Note that, if a point  $Q$  at the intersection of the interface  $\Gamma$  and the boundary falls on an edge with Neumann boundary conditions, then  $Q$  is also included in  $\mathcal{V}_s$ . In order to deal with the singularities arising at the points in  $\mathcal{V}_s$  (which behave differently than the singularities at the other points of  $\mathcal{V}$ ), we also need to augment our weighted Sobolev spaces with a suitable finite-dimensional space. Namely, for each point  $Q \in \mathcal{V}_s$ , we choose a function  $\chi_Q \in \mathcal{C}^\infty(\bar{\Omega})$  that is constant equal to 1 in a neighborhood of  $Q$ . We can choose these functions to have disjoint supports. Let  $W_s$  be the linear span of the functions  $\chi_Q$  for any  $Q \in \mathcal{V}_s$ . We now define the domains and ranges of our operators. Assume first that  $m \geq 1$ .

$$\begin{aligned} \mathcal{D}_{a,m} &:= (\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + W_s) \cap \{u = 0 \text{ on } \partial_D \Omega\} \cap \{u^+ - u^- = 0 \text{ on } \Gamma\} \\ \mathcal{R}_{a,m} &:= \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\Gamma). \end{aligned}$$

Let us observe that, by definition, the functions in  $W_s$  satisfy the interface and boundary conditions (so  $W_s \subset V := H_D^1(\Omega)$ ). Moreover, for  $a \geq 0$ , we have  $\mathcal{D}_{a,m} = (\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + W_s) \cap H_D^1(\Omega)$ . Denote  $A = (a^{ij}, b^i, c) \in \hat{\mathcal{W}}^{m,\infty}(\Omega)^{d^2+d+1}$ ,  $d = 2$ , and  $P^A u = \sum_{i,j=1}^2 \partial_i(a^{ij} \partial_j u) + \sum_{i=1}^2 b^i \partial_i u + cu$ , as before. Then the family of partial differential operators  $\tilde{P}_{a,m}^A : \mathcal{D}_{a,m} \rightarrow \mathcal{R}_{a,m}$

$$\tilde{P}_{a,m}^A u = (Pu, \nabla_\nu^A u|_{\partial_N \Omega}, (\nabla_\nu^A u^+ - \nabla_\nu^A u^-)|_\Gamma) \quad (20)$$

is well defined and the induced map  $\hat{\mathcal{W}}^{m,\infty}(\Omega)^{(d^2+d+1)} \ni A = (a^{ij}, b^i, c) \rightarrow P_{a,m}^A \in \mathcal{L}(\mathcal{D}_{a,m}, \mathcal{R}_{a,m})$  is continuous (recall that  $d = 2$ ). The continuity of this map motivates the use of the spaces  $\hat{\mathcal{W}}^{m,\infty}(\Omega)$ .

When  $m = 0$ , we define

$$\begin{aligned} \mathcal{D}_{a,m} &= \mathcal{D}_{a,0} = \mathcal{K}_{a+1}^1(\Omega) \cap \{u = 0 \text{ on } \partial_D \Omega\} + W_s \\ \mathcal{R}_{a,m} &= \mathcal{R}_{a,0} = (\mathcal{K}_{-a+1}^1(\Omega) \cap \{u = 0 \text{ on } \partial_D \Omega\})^*, \end{aligned}$$

where in the last equation the dual is defined as the dual with pivot  $L^2(\Omega)$ . Then we define  $\tilde{P}_{a,0}$  in a weak sense using the bilinear form  $B$  introduced in Equation (5), as in the smooth case.

Recall the constant  $0 < r$  in the definition of the uniform strict positivity (Definition 1) and Proposition 1. We now state the main result of this section. Recall that  $U$  is reduced to a point in this section.

**Theorem 2.** Assume that  $A = (a^{ij}, b^i, c) \in \hat{\mathcal{W}}^{m,\infty}(\Omega)^{d^2+d+1}$ ,  $d = 2$ , and that  $P^A$  is strictly positive definite on  $V = H_D^1(\Omega)$ . Then there exists  $0 < \eta$  such that for any  $m \in \mathbb{N}_0$  and for any  $0 < a < \eta$ , the map  $\tilde{P}_{a,m}^A : \mathcal{D}_{a,m} \rightarrow \mathcal{R}_{a,m}$  is boundedly invertible and  $\|(\tilde{P}_{a,m}^A)^{-1}\| \leq \tilde{C}$ , where  $\tilde{C} = \tilde{C}(m, r, a, \|A\|_{\hat{\mathcal{W}}^{m,\infty}(\Omega)})$  depends only on the indicated variables.

A more typical formulation is given in the following corollary.

**Corollary 1.** We use the notation and the assumptions of Theorem 2. If  $f \in \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega)$ ,  $g \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)$ , and  $h \in \mathcal{K}_{a-1/2}^{m-1/2}(\Gamma)$ , then the solution  $u \in H_D^1(\Omega)$  of Problem (4) can be written  $u = u_r + u_s$ , with  $u_r \in \hat{\mathcal{K}}_{a+1}^{m+1}(\Omega)$  and  $u \in W_s$ , such that

$$\|u_r\|_{\hat{\mathcal{K}}_{a+1}^{m+1}} + \|u_s\|_{L^2} \leq \tilde{C}(\|f\|_{\hat{\mathcal{K}}_{a-1}^{m-1}} + \|g\|_{\mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)} + \|h\|_{\mathcal{K}_{a-1/2}^{m-1/2}(\Gamma)}),$$

with  $\tilde{C}$  as in Theorem 2.

## 5 Applications

We keep the settings and notations of the previous section. In particular,  $d = 2$  and we are dealing with equations (not systems). One can proceed as in [6,7,13] to obtain  $h^m$ -quasi-optimal rates of convergence for the Galerkin  $u_n$  approximations of  $u$ . Namely, under suitable additional regularity in the  $y \in U$  variable one can construct a sequence of finite dimensional subspaces  $S_n \subset L^2(U; V)$  such that

$$\|u - u_n\|_{L^2(U; V)} \leq C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)}. \quad (21)$$

This is based on a holomorphic regularity in  $U$  and on the approximation properties in [10]. We now state a uniform shift theorem for our families of boundary value/transmission problems.

Let us denote by  $\mathcal{C}_b^k(U; Z)$  the space of  $k$ -times boundedly differentiable functions defined on  $U$  with values in the Banach space  $Z$ . By  $\mathcal{C}_b^\omega(U; Z)$  we shall denote the space of *analytic* functions with bounded derivatives defined on  $U$  with values in the Banach space  $Z$ . Recall that  $r$  is the constant appearing in the definition of uniform positivity of the family  $(P_y)_{y \in U}$ . Theorem 2 extends to families of boundary value problems as in [13] as follows. Let us denote by  $\eta(y)$  the best constant appearing in Theorem 2 for  $P = P_y$  and  $\eta = \inf_{y \in U} \eta(y)$ .

**Theorem 3.** Let  $m \in \mathbb{N}_0$  and  $k_0 \in \mathbb{N}_0 \cup \{\infty, \omega\}$  be fixed. Assume that  $A = (a^{ij}, b^i, c) \in \mathcal{C}_b^{k_0}(U; \hat{\mathcal{W}}^{m,\infty}(\Omega))^{d^2+d+1}$ ,  $d = 2$ , and that the family  $P_y^A$  is uniformly positive definite. Then  $\eta = \inf_{y \in U} \eta(y) > 0$ . Let  $f \in \mathcal{C}_b^{k_0}(U; \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega))$ ,  $g \in \mathcal{C}_b^{k_0}(U; \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega))$ ,  $h \in \mathcal{C}_b^{k_0}(U; \mathcal{K}_{a-1/2}^{m-1/2}(\Gamma))$ , and  $0 < a < \eta$ . Then the solution  $u$  of our family of boundary value problems (4) satisfies  $u \in \mathcal{C}_b^{k_0}(U; \mathcal{D}_{a,m})$ . Moreover, for each finite  $k \leq k_0$ , there exists a constant  $C_{a,m} > 0$  such that

$$\begin{aligned} \|u\|_{\mathcal{C}_b^k(U; \mathcal{D}_{a,m})} &\leq C_{a,m} (\|f\|_{\mathcal{C}_b^k(U; \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega))} \\ &\quad + \|g\|_{\mathcal{C}_b^k(U; \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega))} + \|h\|_{\mathcal{C}_b^k(U; \mathcal{K}_{a-1/2}^{m-1/2}(\Gamma))}). \end{aligned}$$

The constant  $C_{a,m}$  depends only on  $r, m, a, k$ , and the norms of the coefficients  $a^{ij}, b^i, c$  in  $C_b^k(U; \hat{W}^{m,\infty}(\Omega))$ , but not on  $f$  or  $g$ .

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