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# Finite element analysis for the axisymmetric Laplace operator on polygonal domains

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ABSTRACT

Let $\mathcal{L} := -r^{-2}(r\partial_r)^2 - \partial_z^2$. We consider the equation $\mathcal{L}u = f$ on a bounded polygonal domain with suitable boundary conditions, derived from the three-dimensional axisymmetric Poisson's equation. We establish the well-posedness, regularity, and Fredholm results in weighted Sobolev spaces, for possible singular solutions caused by the singular coefficient of the operator $\mathcal{L}$, as $r \to 0$, and by non-smooth points on the boundary of the domain. In particular, our estimates show that there is no loss of regularity of the solution in these weighted Sobolev spaces. Besides, by analyzing the convergence property of the finite element solution, we provide a construction of improved graded meshes, such that the *quasi-optimal* convergence rate can be recovered on piecewise linear functions for singular solutions. The introduction of a new projection operator from the weighted space to the finite element subspace, certain scaling arguments, and a calculation of the index of the Fredholm operator, together with our regularity results, are the ingredients of the finite element estimates.

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## 1. Introduction

Let $\tilde{\Omega} := \Omega \times [0, 2\pi) \subset \mathbb{R}^3$ be a bounded domain, formed by the revolution of the polygon $\Omega \subset \mathbb{R}^2$ with respect to the $z$-axis (see Fig. 1). Consider the three-dimensional Poisson's equation in $\tilde{\Omega}$, with zero Dirichlet boundary conditions. In the presence of axisymmetry in the data, the Laplace operator in the three-dimensional domain becomes the two-dimensional elliptic operator

$$\mathcal{L} := -\frac{1}{r^2}(r\partial_r)^2 - \partial_z^2, \quad r > 0,$$

where $r$ and $z$ are the variables in the cylindrical coordinates $(r, \theta, z)$. Consequently, the three-dimensional axisymmetric Poisson's equation can be reduced to

$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad u|_{\Gamma_0} = 0, \tag{1}$$

where $\Gamma_0 := \partial\tilde{\Omega} \cap \partial\Omega$. We are interested in studying the finite element method (FEM) for the elliptic equation (1). The reduction of the dimension (from three dimensions to two dimensions) leads to substantial savings on the computation of the numerical solution for the original three-dimensional elliptic boundary value problem, and hence is of practical interest.

Suppose the closure of the domain $\Omega$ intersects the $z$-axis. Despite the benefit in numerical computation, this process, however, introduces singular coefficients in the elliptic operator $\mathcal{L}$ and results in Sobolev spaces

$$H_r^m(\Omega) = \{v,\ r^{1/2}\partial_r^i\partial_z^j v \in L^2(\Omega),\ i + j \le m\}$$

with weights vanishing at $r = 0$, which raises difficulties both in the analysis of the equation and in the estimates of the FEM. For the validation on the reduction of the dimension, it is shown in [1,2] that, the three-dimensional Poisson's equation

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is equivalent to the two-dimensional equation (1), by using Fourier analysis to prove certain isomorphisms between the usual Sobolev spaces $H^m(\tilde{\Omega})$ and the weighted spaces $H_r^m(\Omega)$. An approximation property of the finite element solution for the axisymmetric Stokes problem in the space $H_r^m$ is discussed in [3]. We also mention [4,5], in which the Fourier–FEM, a combination of the approximating Fourier and the FEM, is studied for the axisymmetric Poisson's equation. In addition, estimates on the convergence of the multigrid method for the axisymmetric Laplace operator and for the Maxwell operator can be found in [6,7], respectively.

Assuming sufficient regularity of the solution of Eq. (1), the existing results (see [3,6–8] and references therein) suggest that the $H_r^1$-norm of the error between the linear finite element solution and the real solution is bounded by $Ch$ on the triangulation with quasi-uniform triangles of size $h$. This provides the analogy of the *quasi-optimal* convergence rate of the finite element solution for elliptic boundary value problems with regular coefficients in the usual Sobolev spaces and ensures good finite element approximations for the three-dimensional axisymmetric equation with a much lighter computational load than solving the original three-dimensional problem.

Furthermore, the solution of Eq. (1) may have singularities even in these weighted spaces $H_r^m(\Omega)$, due to the non-smooth points on the boundary $\partial\Omega$ and to the singular coefficient when $r \to 0$. The less regularity in the solution slows down the convergence rate of the finite element solution, as well as raises well-posedness concerns in these weighted spaces. Note that near the vertices of $\Omega$ that are not on the $z$-axis, the coefficients of the operator $\mathcal{L}$ are bounded and therefore, the singularities in the solution have the same character as the corner singularities of regular elliptic equations on polygonal domains. There exists a great deal of literature regarding different aspects of corner singularities of two-dimensional elliptic equations. See for example the monographs [9–14], research papers [15–24] on the analysis of the singular solution, and [25–28,16,29–31] and references therein on the numerical approximation for singular solutions of this type. For vertices on the $z$-axis, the situation is different, since the coefficient $1/r \to \infty$. It turns out that the possible singularities near these vertices are closely related to the three-dimensional vertex singularities of elliptic equations. This is our starting point for the work presented in this paper. See [32–36] for discussions on singular solutions of three-dimensional differential equations.

Different from the existing results mentioned above [3,2,6,7,4,5], we shall focus here on establishing well-posedness and regularity results for singular solutions of Eq. (1) in suitable Sobolev spaces and on the construction of simple, explicit finite element schemes to approximate these solutions quasi-optimally. Our goal shall be achieved by introducing the framework in a modified weighted Sobolev space $\mathcal{K}_{a,r}^m(\Omega)$ (Definition 2.7), which allows us to apply certain usual finite element formulations to Eq. (1). In the convergence analysis of the finite element solution, we introduce a new interpolation operator from a local regularization process (Definition 4.4). Compared with the usual nodal interpolation, this regularization technique demonstrates critical properties of functions in the weighted spaces, which are also useful to treat other axisymmetric problems (see [6,37]).

The rest of the paper is organized as follows. In Section 2, we first briefly recall some existing results in the literature for the axisymmetric equation. Then, we define two types of weighed Sobolev spaces for further analysis in Sections 3 and 4, as well as notation that will be used throughout this paper. In addition, several relevant properties of the weighted Sobolev space will be discussed.

In Section 3, we establish our a priori estimates (well-posedness, regularity, and the Fredholm property) for the axisymmetric equation in the weighted space $\mathcal{K}_{a,r}^m(\Omega)$. In particular, we shall show the operator

$$\mathcal{L} : \mathcal{K}_{a+1,+}^2(\Omega) \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}_{a-1,r}^0(\Omega)$$

defines an isomorphism for $a > 0$ small and is Fredholm as long as $a$ is away from a countable set of values. This allows us to compute the range of the index $a$, in which the isomorphism above still holds.

The finite element solution for Eq. (1) is studied in Section 4. In the first part of this section, we briefly present the approximation property of piecewise linear polynomials in the weighted space $H_r^2(\Omega)$. With a new interpolation operator, we show that the *quasi-optimal* convergence rate of the linear finite element solution is attained, assuming the solution is sufficiently regular. Based on these results and on a scaling argument, in the second part of Section 4, we analyze the convergence rate of the numerical solution in the weighted space $\mathcal{K}_{a,r}^m(\Omega)$. Then, we describe a construction of a sequence of triangulations suitably graded to the vertices, such that the *quasi-optimal* rate is recovered for singular solutions.

In Section 5, we present numerical tests for Eq. (1) on two domains for different singularities (on the $z$-axis or away from the $z$-axis). The rates of convergence of the finite element solutions from different meshes are compared. These tests suggest that the *quasi-optimal* convergence rates are achieved on our graded meshes, which is in complete agreement with the theory.

## 2. Weighted Sobolev spaces $H_r^m$ and $\mathcal{K}_{a,r}^m$

In this section, we formally introduce the axisymmetric Poisson's equation and the definitions of some weighted Sobolev spaces with relevant properties.

### 2.1. The axisymmetric Poisson's equation

Let $\tilde{\Omega} := \Omega \times [0, 2\pi) \subset \mathbb{R}^3$ be a bounded domain, which is the revolution of $\Omega$ about the $z$-axis. Suppose $\tilde{\Omega}$ intersects the $z$-axis and its half section (the intersection of $\tilde{\Omega}$ and a meridian half plane) $\Omega \subset \mathbb{R}^2$ is a polygon (see, for example, Fig. 1).

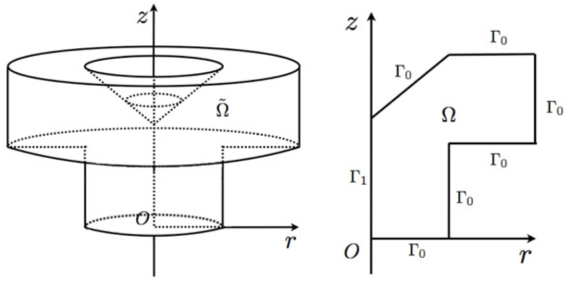


**Fig. 1.** A three-dimensional axisymmetric domain $\tilde{\Omega}$ (left); the corresponding two-dimensional half section $\Omega$ (right).

We then consider the three-dimensional Poisson's equation in $\tilde{\Omega}$ with the Dirichlet boundary condition,

$$-\Delta\tilde{u} = -(\partial_x^2 + \partial_y^2 + \partial_z^2)\tilde{u} = \tilde{f} \quad \text{in } \tilde{\Omega}, \qquad \tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega}. \tag{2}$$

Recall the Sobolev space $H^m(\tilde{\Omega}) = \{v, \partial^\alpha v \in L^2(\tilde{\Omega}), |\alpha| \le m\}$ and $H_0^1(\tilde{\Omega}) := H^1(\tilde{\Omega}) \cap \{v|_{\partial\tilde{\Omega}} = 0\}$ in the trace sense, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index.

In the presence of axisymmetry in the data and in the solution ($\partial_\theta\tilde{u} = \partial_\theta\tilde{f} = 0$) with respect to the cylindrical coordinates $(r, \theta, z)$, we define

$$u(r, z) = \tilde{u}(r, \theta, z) \quad \text{and} \quad f(r, z) = \tilde{f}(r, \theta, z). \tag{3}$$

Recall $\Omega$ is the intersection of $\tilde{\Omega}$ and the $rz$-plane for $r > 0$. We let $\Gamma_0 := \partial\tilde{\Omega} \cap \partial\Omega$ be the part of the boundary $\partial\Omega$ imposed with the Dirichlet condition and $\Gamma_1 := \partial\Omega \setminus \Gamma_0$ be its complement set on the $z$-axis (Fig. 1). Thus, it is well known that Eq. (2) can be written as the following elliptic equation (see also [1,7]),

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \Gamma_0, \end{cases} \tag{4}$$

where

$$\mathcal{L} := -\frac{1}{r^2}(r\partial_r)^2 - \partial_z^2 = -\partial_r^2 - r^{-1}\partial_r - \partial_z^2.$$

Note that the derivation of the weak solution of Eq. (4) also requires boundary conditions on $\Gamma_1$, which we will discuss in Remark 2.5. From now on, we shall concentrate on the analysis of Eq. (4) and its finite element approximations. Our techniques, nevertheless, may be also useful for dealing with different boundary conditions and other types of axisymmetric problems.

### *2.2. Weighted Sobolev spaces and the weak solution*

We here define different weighted Sobolev spaces on $\Omega$ and the weak solution of Eq. (4), with which further analysis can be carried out in Sections 3 and 4.

**Definition 2.1.** We first define the following weighted Sobolev spaces

$$L_r^2(\Omega) := \left\{ v : \Omega \to \mathbb{R}, \ \int_\Omega v^2 r dr dz < \infty \right\},$$

$$H_r^m(\Omega) := \{v : \Omega \to \mathbb{R}, \partial_r^i\partial_z^j v \in L_r^2(\Omega), \ 0 \le i + j \le m\}, \quad m = 0, 1, 2.$$

Clearly, $H_r^0(\Omega) = L_r^2(\Omega)$. Then, the norms and the semi-norms for any $v \in H_r^m(\Omega)$, $m = 0, 1, 2$, are defined by

$$\|v\|_{H_r^m(\Omega)}^2 := \sum_{i+j=0}^{m} \int_\Omega (\partial_r^i\partial_z^j v)^2 r dr dz, \qquad |v|_{H_r^m(\Omega)}^2 := \sum_{i+j=m} \int_\Omega (\partial_r^i\partial_z^j v)^2 r dr dz.$$

Note that $H_r^m(\Omega)$ is closely related to the usual Sobolev space $H^m(\tilde{\Omega})$, $m = 0, 1, 2$. In particular, we summarize a number of results from the literature for a better understanding of the space $H_r^m(\Omega)$. Let $\breve{H}^m(\tilde{\Omega}) \subset H^m(\tilde{\Omega})$, $m = 0, 1, 2$, be the subspace of all axisymmetric functions in $H^m(\tilde{\Omega})$. The following two propositions can be found in [1,2].

**Proposition 2.2.** *For $m = 0, 1$, the trace mapping $\tilde{v}(r, \theta, z) \to v(r, z)$ in (3), is well defined for smooth functions and extends to an isometry from $\breve{H}^m(\tilde{\Omega})$ onto $H_r^m(\Omega)$. The reciprocal lifting, $v \in H_r^m(\Omega) \to \tilde{v} \in \breve{H}^m(\tilde{\Omega})$ also defines an isometry.*

$$2\pi\|v\|_{H_r^m(\Omega)}^2 = \|\tilde{v}\|_{H^m(\tilde{\Omega})}^2.$$

On the other hand, for $m = 2$, we have the following proposition.

**Proposition 2.3.** *Let* $H^2_+(\Omega) := \{v \in H^2_r(\Omega),\ \partial_r v/r \in L^2_r(\Omega)\}$, *with*

$$\|v\|_{H^2_+(\Omega)} = \left(\|v\|^2_{H^2_r(\Omega)} + \int_\Omega \frac{(\partial_r v)^2}{r} drdz\right)^{1/2}. \tag{5}$$

*The trace operator in Proposition 2.2 defines an isomorphism from* $\breve{H}^2(\tilde{\Omega})$ *to* $H^2_+(\Omega)$.

Besides, the following density property can be found in [38] (Proposition 7.6).

**Proposition 2.4.** *For* $m = 0, 1, 2$, *the space of smooth functions* $\mathcal{C}^\infty(\bar{\Omega})$ *is dense in* $H^m_r(\Omega)$.

**Remark 2.5.** It is shown in [1], that for any $u \in H^2_+(\Omega)$, the trace $u \to \partial_r u|_{\Gamma_1}$ is well defined, and $\partial_r u|_{\Gamma_1} = 0$ in $L^2$. Therefore, if the solution $u$ belongs to $H^2_+(\Omega) \cap \{v|_{\Gamma_0} = 0\}$, with integration by parts, we define the weak solution $u \in H^1_r(\Omega) \cap \{v|_{\Gamma_0} = 0\}$ of Eq. (4) by

$$a_w(u, v) := \int_\Omega (\partial_r u \partial_r v + \partial_z u \partial_z v) r drdz = \int_\Omega f v r drdz, \tag{6}$$

for any $v \in H^1_r(\Omega) \cap \{v|_{\Gamma_0} = 0\}$. By the Poincaré inequality on the three-dimensional domain $\tilde{\Omega}$ and the Lax–Milgram Lemma, it is seen that the weak form (6) determines a unique solution $u \in H^1_r(\Omega) \cap \{v|_{\Gamma_0} = 0\}$, for any $f \in L^2_r(\Omega)$. In fact, one can further show $u \in H^2_+(G)$, for any $G \subset \Omega$ away from the vertices, based on the standard regularity estimates for Eq. (2). In addition, Proposition 2.2 implies that $\tilde{u}(r, \theta, z) = u(r, z)$ is the weak solution of the original Poisson's Eq. (2), with $\tilde{f}(r, \theta, z) = f(r, z) \in L^2(\tilde{\Omega})$ (see also [1,7] and references within).

We write a few words about the trace on $\Gamma_0$ for any function in $H^1_r(\Omega)$. Note that $\Gamma_0$ is composed of line segments $\gamma_i$. Then, on a segment $\gamma_i \subset \Gamma_0$, $\bar{\gamma}_i \cap \{r = 0\} = \emptyset$, the trace of $v \in H^1_r(\Omega)$ is well defined in $L^2$, because in the neighborhood of $\gamma_i$, $H^1_r$ is equivalent to $H^1$. For a segment $\gamma_i$ whose closure intersects the $z$-axis, we recall the following result from [6].

**Proposition 2.6.** *Let* $T \subset \Omega$ *be a triangle with diameter* $h$, *such that* $\bar{T} \cap \{r = 0\} \neq \emptyset$. *Let* $e$ *be an edge of* $T$, *not sitting on the* $z$-*axis, but* $\bar{e} \cap \{r = 0\} \neq \emptyset$. *Then, for any* $v \in \mathcal{C}^\infty(\bar{T})$,

*Case* 1: *if* $\bar{T} \cap \{r = 0\}$ *is only a point, then*

$$\int_e r^2 v^2 ds \le C|e|h(h^{-2}\|v\|^2_{L^2_r(T)} + |v|^2_{H^1_r(T)});$$

*Case* 2: *if* $\bar{T} \cap \{r = 0\}$ *is an edge, then*

$$\|v\|^2_{L^2_r(e)} \le C|e|(h^{-2}\|v\|^2_{L^2_r(T)} + |v|^2_{H^1_r(T)}).$$

*The constant* $C$ *above depends on the shape regularity of the triangle, not on* $v$. *The extension of these inequalities for* $v \in H^1_r(\Omega)$ *follows from the density argument in Proposition 2.4.*

Thus, for any $v \in H^1_r(\Omega)$, if $\bar{\gamma}_i \cap \{r = 0\}$ is an isolated point in $\partial\Omega \cap \{r = 0\}$, $\gamma_i$ can be included in a triangle of Case 1, and hence $\int_{\gamma_i} r^2 v^2 drdz$ is well defined; if $\bar{\gamma}_i \cap \{r = 0\}$, however, is an end point of a segment in $\partial\Omega \cap \{r = 0\}$, then $v$ has a $L^2_r$-trace on $\gamma_i$, since $\gamma_i$ can be a edge of a triangle of Case 2. This trace result will also be useful in our finite element analysis in Section 4.

Recall that the solution of Eq. (4) may have singularities in $H^2_+(\Omega)$, due to the non-smooth points on the boundary and to the singular coefficient in $\mathcal{L}$, even if the right hand side $f$ is smooth. To handle these possible singular solutions, we need the following weighted Sobolev space.

Let $Q_i$ be the $i$th vertex of $\Omega$ and $\mathcal{S} = \{Q_i\}$ be its vertex set. Denote by $l$ the minimum of the non-zero distances from a point $Q_i$ to a boundary edge of $\Omega$. Let

$$\tilde{l} := \min(1/2, l/4) \quad \text{and} \quad \mathcal{V}_i := \Omega \cap B(Q_i, \tilde{l}), \tag{7}$$

where $B(Q_i, \tilde{l})$ denotes the ball centered at $Q_i \in \mathcal{S}$ with radius $\tilde{l}$. Note that the sets $\mathcal{V}_i$ are disjoint. Then, we define the function $\vartheta \in \mathcal{C}^\infty(\bar{\Omega} \setminus \mathcal{S})$

$$\vartheta(x) \begin{cases} = |x - Q_i| & \text{in } \mathcal{V}_i \\ \ge \tilde{l}/2 & \text{in } \Omega \setminus (\cup \mathcal{V}_i). \end{cases} \tag{8}$$

Thus, the space $\mathcal{K}^m_{a,r}(\Omega)$ is given by the definition below.

**Definition 2.7.** We define

$$\mathcal{K}^m_{a,r}(\Omega) := \{v : \Omega \to \mathbb{R},\ \vartheta^{i+j-a}\partial^i_r\partial^j_z v \in L^2_r(\Omega),\ \forall i+j \le m\}, \quad a \in \mathbb{R},\ m = 0, 1, 2.$$

For any open set $G \subseteq \Omega$ and any $v : G \to \mathbb{R}$,

$$\|v\|^2_{\mathcal{K}^m_{a,r}(G)} := \sum_{i+j=0}^{m} \|\vartheta^{i+j-a}\partial^i_r\partial^j_z v\|^2_{L^2_r(G)}, \qquad |v|^2_{\mathcal{K}^m_{a,r}(G)} := \sum_{i+j=m} \|\vartheta^{m-a}\partial^i_r\partial^j_z v\|^2_{L^2_r(G)}.$$

The inner product on the Hilbert space $\mathcal{K}^m_{a,r}(\Omega)$ is

$$(u, v)_{\mathcal{K}^m_{a,r}(\Omega)} = \sum_{i+j\le m} \int_\Omega \vartheta^{2(i+j-a)}(\partial^i_r\partial^j_z u)(\partial^i_r\partial^j_z v) r dr dz.$$

A subspace of $\mathcal{K}^2_{a,r}(\Omega)$ that can be regarded as the counterpart of $H^2_+(\Omega)$ is

$$\mathcal{K}^2_{a,+}(\Omega) := \mathcal{K}^2_{a,r}(\Omega) \cap \left\{v,\ \int_\Omega \vartheta^{4-2a}\frac{(\partial_r v)^2}{r} dr dz < \infty\right\},$$

with the norm on any open set $G \subset \Omega$,

$$\|v\|^2_{\mathcal{K}^2_{a,+}(G)} = \|v\|^2_{\mathcal{K}^1_{a,r}(G)} + |v|^2_{\mathcal{K}^2_{a,+}(G)},$$

where

$$|v|^2_{\mathcal{K}^2_{a,+}(G)} = \int_G \vartheta^{4-2a}\left((\partial^2_r v)^2 + (\partial^2_z v)^2 + (\partial_r\partial_z v)^2 + \left(\frac{\partial_r v}{r}\right)^2\right) r dr dz.$$

In addition, we denote by $\mathcal{K}^{-1}_{a,r}(\Omega) := (\mathcal{K}^1_{a,r}(\Omega) \cap \{v|_{\Gamma_0} = 0\})'$ the dual space of $\mathcal{K}^1_{a,r}(\Omega) \cap \{v|_{\Gamma_0} = 0\}$ with respect to the pivot space $L^2_r(\Omega)$,

$$\|w\|_{\mathcal{K}^{-1}_{a,r}(\Omega)} := \sup_{v\in\mathcal{K}^1_{a,r}(\Omega)\cap\{v|_{\Gamma_0}=0\}} \frac{|\int_\Omega vwrdrdz|}{\|v\|_{\mathcal{K}^1_{a,r}(\Omega)}}, \quad v \ne 0.$$

**Remark 2.8.** The space $\mathcal{K}^m_{a,r}(\Omega)$ for Eq. (4) is the analogue of the weighted space $\mathcal{K}^m_a(\Omega)$ for corner singularities of elliptic equations with bounded coefficients (see for example [16,21,39]). In the definitions of weighted spaces $H^m_r(\Omega)$ and $\mathcal{K}^m_{a,r}(\Omega)$, we consider only for $m = 0, 1, 2$. This is sufficient for our FEM using linear approximation functions. An extension for $m > 2$ is also possible [2]. By Proposition 2.2, it is natural to introduce $H^m_r(\Omega)$ if the solution is regular enough. On the other hand, we shall show that $\mathcal{K}^m_{a,r}(\Omega)$ is an appropriate space to study singular solutions from the singular coefficients and from the non-smooth boundary for Eq. (4).

### *2.3. Some lemmas*

We give several properties of $\mathcal{K}^m_{a,r}(\Omega)$ that are useful for further analysis. To avoid any confusion on notation, in the text below, we use $\rho$ and $\phi$ as the variables in the polar coordinates $(\rho, \phi)$, since the variable $r$ is used in the equation, where $\rho$ denotes the distance to the origin and $\phi$ is the angle. In addition, by $A \simeq B$, we mean that there exist constants $C_1, C_2 > 0$, such that $C_1 A \le B \le C_2 A$. For simplicity, we write $\mathcal{K}^m_{a,r} := \mathcal{K}^m_{a,r}(\Omega)$ and $H^m_r := H^m_r(\Omega)$.

In the following lemmas, we will omit the proof if it is mainly based on definitions of the norms and direct calculation. We first have the following alternative expressions for the operators $\partial_r$ and $\partial_z$.

**Lemma 2.9.** *On every $\mathcal{V}_i := \Omega \cap B(Q_i, \tilde{l})$, we set a local polar coordinate system $(\rho, \phi)$, where $Q_i = (r_i, z_i)$ is the new origin, and $r - r_i = \rho\sin\phi$, $z - z_i = \rho\cos\phi$. Then, on $\mathcal{V}_i$,*

$$\partial_r = (\sin\phi)\partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi, \qquad \partial_z = (\cos\phi)\,\partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi.$$

*Meanwhile, the relation between the Cartesian coordinates $(x, y, z)$ and the cylindrical coordinates $(r, \theta, z)$ reads*

$$\partial_x = (\cos\theta)\,\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_y = (\sin\theta)\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

Recall the function $\vartheta$ in (8). Then, we have an upper bound for the following function.

**Lemma 2.10.** *The function $\vartheta^{j+k-a}\partial^j_r\partial^k_z\vartheta^a$ is bounded on $\Omega$.*

This lemma leads to the following isomorphism between weighted Sobolev spaces.

**Lemma 2.11.** *We have $\vartheta^b\mathcal{K}^m_{a,r} = \mathcal{K}^m_{a+b,r}$, where $\vartheta^b\mathcal{K}^m_{a,r} = \{\vartheta^b v,\ v \in \mathcal{K}^m_{a,r}\}$.*

**Proof.** Let $v \in \mathcal{K}^m_{a,r}$ and $w = \vartheta^b v$. Then $|\vartheta^{i+j-a}\partial_r^i\partial_z^j v| \in L^2_r$, for $i+j \leq m$. Thus, we verify $w \in \mathcal{K}^m_{a+b,r}$ by checking the inequalities below,

$$|\vartheta^{i+j-a-b}\partial_r^i\partial_z^j w| = \left|\vartheta^{i+j-a-b}\sum_{s\leq i,t\leq j}\binom{i}{s}\binom{j}{t}\partial_r^s\partial_z^t\vartheta^b\partial_r^{i-s}\partial_z^{j-t}v\right|$$
$$\leq C\sum_{s\leq i,t\leq j}\left|\vartheta^{(i+j-s-t)-a}\partial_r^{i-s}\partial_z^{j-t}v\right| \in L^2_r,$$

where the last inequality follows from Lemma 2.10. Therefore, $\vartheta^b\mathcal{K}^m_{a,r}$ is continuously embedded in $\mathcal{K}^m_{a+b,r}$. Namely, the map $\vartheta^b : \mathcal{K}^m_{a,r} \to \mathcal{K}^m_{a+b,r}$ is continuous.

On the other hand, because this embedding holds for any real number $b$, we have

$$\mathcal{K}^m_{a+b,r} = \vartheta^b\vartheta^{-b}\mathcal{K}^m_{a+b,r} \subset \vartheta^b\mathcal{K}^m_{a,r}.$$

To complete the proof, we also notice that the inverse of multiplication by $\vartheta^b$ is multiplication by $\vartheta^{-b}$, which is also continuous. □

Recall that $\mathcal{V}_i := \Omega \cap B(Q_i, \tilde{l})$ in (7). Therefore, $\vartheta(r,z) \leq \tilde{l}$ on $\mathcal{V}_i$, and we have the following lemma.

**Lemma 2.12.** *Let $G \subset \mathcal{V}_i$ be an open subset of $\mathcal{V}_i$, such that $\vartheta \leq \xi \leq \tilde{l}$ on $G$. Then, for $m' \leq m$ and $a' \leq a$, we have*

$$\mathcal{K}^m_{a,r} \subset \mathcal{K}^{m'}_{a',r} \quad \text{and} \quad \|v\|_{\mathcal{K}^{m'}_{a',r}(G)} \leq \xi^{a-a'}\|v\|_{\mathcal{K}^m_{a,r}(G)}, \quad \forall v \in \mathcal{K}^m_{a,r}.$$

The following lemma asserts that the $H^m_r$-norm and the $\mathcal{K}^m_{a,r}$-norm are equivalent on a subset of $\Omega$, whose closure is away from the vertex set $\mathcal{S}$.

**Lemma 2.13.** *Let $G \subset \Omega$ be an open subset, such that $\inf_{x\in G}\vartheta(x) > 0$. Then, $\|v\|_{H^m_r(G)} \leq M_1\|v\|_{\mathcal{K}^m_{a,r}(G)}$ and $\|v\|_{\mathcal{K}^m_{a,r}(G)} \leq M_2\|v\|_{H^m_r(G)}$, $\forall v \in H^m_r(G)$. In addition, $\|v\|_{H^2_+(G)} \leq M_1\|u\|_{\mathcal{K}^2_{a,+}(G)}$ and $\|v\|_{\mathcal{K}^2_{a,+}(G)} \leq M_2\|v\|_{H^2_+(G)}$, $\forall v \in H^2_+(G)$, where $M_1$ and $M_2$ depend on the infimum of $\vartheta(x)$ on $G$ and $m$, but not on $v$.*

Using Lemma 2.12, we have the following comparison for $\mathcal{K}^m_{a,r}(\mathcal{V}_i)$ and $H^m_r(\mathcal{V}_i)$.

**Lemma 2.14.** *Let $G \subset \mathcal{V}_i$ be an open subset, on which $\vartheta \leq \xi \leq \tilde{l}$. For $m = 0, 1, 2$,*

$$\|v\|_{H^m_r(G)} \leq \xi^{a-m}\|v\|_{\mathcal{K}^m_{a,r}(G)}, \quad \forall a \geq m; \qquad \|v\|_{\mathcal{K}^m_{a,r}(G)} \leq \xi^{-a}\|v\|_{H^m_r(G)}, \quad \forall a \leq 0.$$

By Lemma 2.13, we have the extension of Lemma 2.14 to the entire domain $\Omega$.

**Corollary 2.15.** *For a function $v$, we have $\|v\|_{H^m_r} \leq M_1\|v\|_{\mathcal{K}^m_{m,r}}$ and $\|v\|_{\mathcal{K}^m_{a,r}} \leq M_2\|v\|_{H^m_r}$ for $a \leq 0$, where $M_1$ and $M_2$ depend on $m$ and $a$.*

Recall the operator $\mathcal{L} := -\partial_r^2 - r^{-1}\partial_r - \partial_z^2$.

**Lemma 2.16.** *The map $\mathcal{L} : \mathcal{K}^2_{a+1,+} \to \mathcal{K}^0_{a-1,r}$ is well defined and continuous.*

**Proof.** We show that there is $C > 0$ such that $\|\mathcal{L}v\|_{\mathcal{K}^0_{a-1,r}} \leq C\|v\|_{\mathcal{K}^2_{a+1,+}}$ for all $v \in \mathcal{K}^2_{a+1,+}$. The proof is then completed by the calculation below.

$$\|\mathcal{L}v\|^2_{\mathcal{K}^0_{a-1,r}} = \int_\Omega \vartheta^{2-2a}(v_{rr} + r^{-1}v_r + v_{zz})^2 rdrdz$$
$$\leq C\int_\Omega \vartheta^{2-2a}(v_{rr}^2 + r^{-2}v_r^2 + v_{zz}^2 + v_{rz}^2)rdrdz$$
$$\leq C\|v\|^2_{\mathcal{K}^2_{a+1,+}}. \quad □$$

## 3. Well-posedness and regularity in weighted Sobolev spaces

Based on the relation between Eq. (4) and the three-dimensional Poisson's equation (2), it can be shown that the solution of (4) $u \in H^2_+ \subset H^2_r$, provided that $f \in L^2_r$ and $\Omega$ has only "good" corners. See [7] for the case $\Omega = (0,1) \times (0,1)$.

On an arbitrary polygonal domain $\Omega$, however, the non-smooth boundary and the singular coefficients of the elliptic operator may affect the well-posedness and regularity of the solution in $H^m_r$, and the statements above are in general not true.

In this section, we analyze the solution of Eq. (4) in the weighted space $\mathcal{K}^m_{a,r} := \mathcal{K}^m_{a,r}(\Omega)$ on polygonal domains. To be more precise, we look for a minimal regularity solution $u \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$ that satisfies the variational formulation

$$a(u, v) = \int_\Omega (\partial_r u \partial_r v + \partial_z u \partial_z v) r dr dz = \int_\Omega f v r dr dz, \tag{9}$$

for any $v \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$. We first prove the well-posedness of the solution in $\mathcal{K}^1_{a+1,r} \cap \{v|_{\Gamma_0} = 0\}$, for $a > 0$ small. Then, we provide a regularity result in the weighed Sobolev space for the solution. The Fredholm property of the operator $\mathcal{L}$ will be discussed in the last subsection.

Throughout Section 3, we denote by $\nabla = (\partial_x, \partial_y, \partial_z)$, the gradient in the conventional Cartesian coordinates.

### 3.1. Well-posedness

We first need the following lemma for our well-posedness result in Theorem 3.2.

**Lemma 3.1.** *For any $u \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$, we have*

$$a(u, u) = \int_\Omega \big[(\partial_r u)^2 + (\partial_z u)^2\big] r dr dz \ge C \int_\Omega \frac{u^2}{\vartheta^2} r dr dz,$$

*and therefore the bilinear form $a(\cdot, \cdot)$ in (9) is strictly coercive on $\mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$.*

**Proof.** Recall the neighborhood $\mathcal{V}_i$ of $Q_i \in \mathcal{S}$. Define $\mathcal{V}_i/\alpha = B(Q_i, \tilde{l}/\alpha) \cap \Omega$ for $\alpha \in \mathbb{N}$ and let $\mathcal{O} = \cup \mathcal{V}_i/2$. By the definitions of the norms involved, we need to verify the following Poincaré-type inequality on every $\mathcal{V}_i$, and also on $\Omega \setminus \mathcal{O}$,

$$\int_\Omega \big[(\partial_r u)^2 + (\partial_z u)^2\big] r dr dz \ge C \int_D \frac{u^2}{\vartheta^2} r dr dz, \tag{10}$$

where $D$ is either $\mathcal{V}_i$ or $\Omega \setminus \mathcal{O}$ and $C$ is independent of $u$. Noting $u \in \mathcal{K}^1_{1,r} \subset H^1_r$, we let $\tilde{u}(r, \theta, z) = u(r, z)$ be the axisymmetric function in the three-dimensional domain as in Proposition 2.2.

For $D = \mathcal{V}_i$, the neighborhood of $Q_i$ that is away from the $z$-axis, note that the desired estimate is well known in [32,12,13] for weighted spaces without the parameter $r$. Thus, the justification of (10) for this sub-domain follows, since $r$ is bounded on $D$ and $\mathcal{K}^1_{1,r}$ is equivalent to the space in the above references.

We now verify (10) for $D = \mathcal{V}_i$, the neighborhood of the vertex $Q_i$ sitting on the $z$-axis. Recall $\tilde{u}(r, \theta, z) = u(r, z)$. Let $\tilde{\Omega}_{\mathcal{V}_i} = \mathcal{V}_i \times [0, 2\pi) \subset \tilde{\Omega}$ be the domain from the revolution of $\mathcal{V}_i$ about the $z$-axis. Thus, $\tilde{\Omega}_{\mathcal{V}_i}$ can be characterized in the spherical coordinates $(\rho, \theta, \phi)$ centered at $Q_i$ by

$$\tilde{\Omega}_{\mathcal{V}_i} = \{(\rho, \omega),\ 0 < \rho < \tilde{l},\ \omega \in \omega_{Q_i}\},$$

where $\omega_{Q_i} \subset S^2$ is the polygonal domain on the unit sphere $S^2$. Then, by Lemma 2.9, we have

$$|\nabla \tilde{u}|^2 = \tilde{u}_x^2 + \tilde{u}_y^2 + \tilde{u}_z^2 = \tilde{u}_\rho^2 + \frac{\tilde{u}_\phi^2}{\rho^2} + \frac{\tilde{u}_\theta^2}{\rho^2 \sin^2 \phi},$$

and

$$\int_{\omega_{Q_i}} \tilde{u}^2 dS \le C \int_{\omega_{Q_i}} \left(\tilde{u}_\phi^2 + \frac{\tilde{u}_\theta^2}{\sin^2 \phi}\right) \sin \phi d\phi d\theta,$$

which is just the Poincaré inequality on $\omega_{Q_i}$ and $dS = \sin \phi d\phi d\theta$ is the volume element on $\omega_{Q_i}$ (see also [32,33]). Thus, we obtain

$$\begin{aligned} 2\pi \int_{\mathcal{V}_i} \frac{u^2}{\vartheta^2} r dr dz &= \int_{\tilde{\Omega}_{\mathcal{V}_i}} \frac{\tilde{u}^2}{\rho^2} dx dy dz = \int_0^{\tilde{l}} \int_{\omega_{Q_i}} \tilde{u}^2 dS d\rho \\ &\le C \int_0^{\tilde{l}} \int_{\omega_{Q_i}} \Big(\tilde{u}_\rho^2 + \frac{\tilde{u}_\phi^2}{\rho^2} + \frac{\tilde{u}_\theta^2}{\rho^2 \sin^2 \phi}\Big) \rho^2 dS d\rho \\ &= C \int_{\tilde{\Omega}_{\mathcal{V}_i}} |\nabla \tilde{u}|^2 dx dy dz = 2\pi C \int_{\mathcal{V}_i} (u_r^2 + u_z^2) r dr dz. \end{aligned}$$

We now verify for $D = \Omega \setminus \mathcal{O}$. Recall $\tilde{\Omega} = \Omega \times [0, 2\pi)$. Then, we have

$$\begin{aligned} 2\pi \int_\Omega \big[(\partial_r u)^2 + (\partial_z u)^2\big] r dr dz &= \int_0^{2\pi} \int_\Omega \big[(\partial_r \tilde{u})^2 + (\partial_z \tilde{u})^2\big] r dr dz d\theta \\ = \int_{\tilde{\Omega}} |\nabla \tilde{u}|^2 dx dy dz \ge C \int_{\tilde{\Omega}} \tilde{u}^2 dx dy dz &= 2\pi C \int_\Omega u^2 r dr dz \ge 2\pi C \int_D \vartheta^{-2} u^2 r dr dz, \end{aligned}$$

where we applied the equation $|\nabla\tilde{u}|^2 = \tilde{u}_r^2 + (\tilde{u}_\theta/r)^2 + \tilde{u}_z^2$, the usual Poincaré inequality on $\tilde{\Omega}$, and the fact that $\vartheta$ is bounded from 0 on $\Omega \setminus \mathcal{O}$.

Adding all the inequalities, we actually show that inequality (10) holds on $\Omega$, and hence complete the proof. $\square$

Based on Lemma 3.1, we note that the two spaces $\mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$ and $H^1_r \cap \{v|_{\Gamma_0} = 0\}$ are essentially the same, since they can be equipped with the same norm $a(\cdot,\cdot)^{1/2}$. Therefore, the $\mathcal{K}^1_1$-weak solution from (9) is the same solution defined by (6) in Section 2. We now have the solvability result in the space $\mathcal{K}^1_{a+1,r}$.

**Theorem 3.2.** *There exists $\eta > 0$, such that for $0 \le a < \eta$ and $f \in \mathcal{K}^0_{a-1,r}$, the variational formulation (9) defines a unique solution $u \in \mathcal{K}^1_{a+1,r}$ of Eq. (4).*

**Proof.** We first verify, for $a = 0$, the uniqueness of the solution $u \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$. Note that the Cauchy–Schwarz inequality gives

$$a(u, v) \le |u|_{\mathcal{K}^1_{1,r}} |v|_{\mathcal{K}^1_{1,r}} \le \|u\|_{\mathcal{K}^1_{1,r}} \|v\|_{\mathcal{K}^1_{1,r}}.$$

Based on this continuity property and Lemma 3.1, the Lax–Milgram Lemma then proves that

$$\mathcal{L} : \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^{-1}_{1,r} := (\mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\})'$$

is an isomorphism. Therefore, we conclude that there exists a unique solution $u \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$ for $f \in \mathcal{K}^0_{a-1,r} \subset \mathcal{K}^{-1}_{1,r}$.

For $a \ge 0$, note that the family of operators

$$\vartheta^{-a}\mathcal{L}\vartheta^a : \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^{-1}_{1,r}$$

depends on $a$ continuously in norm. Therefore, there exists $\eta > 0$, depending on the domain and the operator $\mathcal{L}$, such that for $0 \le a < \eta$, the operator

$$\vartheta^{-a}\mathcal{L}\vartheta^a : \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^{-1}_{1,r}$$

is invertible.

Hence, by Lemma 2.11, for $0 \le a < \eta$, since $\mathcal{K}^0_{-1,r} \subset \mathcal{K}^{-1}_{1,r}$, the invertibility of $\vartheta^{-a}\mathcal{L}\vartheta^a : \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^{-1}_{1,r}$ proves the solution $u \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$ is in fact a solution in $\mathcal{K}^1_{a+1,r} \cap \{v|_{\Gamma_0} = 0\}$, for any $f \in \mathcal{K}^0_{a-1,r} = \vartheta^a\mathcal{K}^0_{-1,r}$. $\square$

See also [16,39–41] for related discussions. The parameter $\eta$ plays an important role in our analysis of the FEM in Section 4. By computing the index of $\mathcal{L}$ in the weighted Sobolev space $\mathcal{K}^m_{a+1,r}$, we shall evaluate $\eta$ explicitly in Theorem 3.5 using the Fredholm property of the operator.

### 3.2. Regularity

Based on the regularity estimates in [32,16,33,21,39] for the Laplace operator in two-dimensional polygonal and three-dimensional polyhedral domains, we now have the following regularity result for the solution of the axisymmetric boundary value problem (4), in the weighted Sobolev space $\mathcal{K}^m_{a,r}$.

**Theorem 3.3.** *Let $0 \le a < 1$ and $f \in L^2_r$. Suppose $u \in \mathcal{K}^1_{a+1,r}$ is the unique solution of Eq. (4). Then, we have*

$$\|u\|_{\mathcal{K}^2_{a+1,+}} \le C\|f\|_{L^2_r},$$

*where the constant $C = C(a, \Omega) > 0$ is independent of $f$ and $u$.*

**Proof.** Note that for $0 \le a < 1$, $f \in L^2_r \subset \mathcal{K}^0_{a-1,r}$. By Lemma 3.1 and the definitions of weighted spaces, $u \in \mathcal{K}^1_{a+1,r} \subset H^1_r$. We let $\tilde{u}(r,\theta,z) := u(r,z)$ and $\tilde{f}(r,\theta,z) := f(r,z)$ as in Proposition 2.2. Then, $\tilde{u} \in H^1(\tilde{\Omega})$ solves the three-dimensional Poisson's equation (2) with the right hand side $\tilde{f} \in L^2(\tilde{\Omega})$. According to Theorem 3.2, $\vartheta^{-a}\mathcal{L}\vartheta^a : \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^{-1}_{1,r}$ is invertible. Thus,

$$\|\vartheta^{-a}u\|_{\mathcal{K}^1_{1,r}} \le C\|\vartheta^{-a}f\|_{\mathcal{K}^{-1}_{1,r}} = C \sup_{0 \ne w \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0}=0\}} \frac{\int_\Omega \vartheta^{-a} f w r dr dz}{\|w\|_{\mathcal{K}^1_{1,r}}}.$$

For $f \in L^2_r \subset \mathcal{K}^0_{a-1,r}$, based on Lemma 2.11, the Cauchy–Schwarz inequality, and the estimate above, we have

$$\|u\|_{\mathcal{K}^1_{a+1,r}} \le C \sup_{0 \ne w \in \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0}=0\}} \frac{\|f\|_{\mathcal{K}^0_{a-1,r}} \|w\|_{\mathcal{K}^0_{1,r}}}{\|w\|_{\mathcal{K}^1_{1,r}}} \le C\|f\|_{\mathcal{K}^0_{a-1,r}} \le C\|f\|_{L^2_r}. \tag{11}$$

Therefore, we only need to verify $|u|_{\mathcal{K}^2_{a+1,+}} \le C\|f\|_{L^2_r}$.

Recall $\mathcal{V}_i = B(Q_i, \tilde{l}) \cap \Omega$ in (7) and $\mathcal{V}_i/\alpha = B(Q_i, \tilde{l}/\alpha) \cap \Omega$ for $\alpha \in \mathbb{N}$. Regularity is a local property. We prove this estimate on each $\mathcal{V}_i/2$ and on $\Omega \setminus \mathcal{O} := \Omega \setminus (\cup \mathcal{V}_i/2)$, respectively.

Let $\tilde{\Omega}_{-\mathcal{O}} := (\Omega \setminus \mathcal{O}) \times [0, 2\pi) \subset \tilde{\Omega}$ (resp. $\tilde{\Omega}_{-\mathcal{P}} := \big(\Omega \setminus (\cup \mathcal{V}_i/4)\big) \times [0, 2\pi) \subset \tilde{\Omega}$) be obtained by the revolution of $\Omega \setminus \mathcal{O}$ (resp. $\Omega \setminus (\cup \mathcal{V}_i/4)$) about the $z$-axis. By the standard regularity result for Eq. (2), we have

$$|\tilde{u}|_{H^2(\tilde{\Omega}_{-\mathcal{O}})} \le C(\|\tilde{f}\|_{L^2(\tilde{\Omega}_{-\mathcal{P}})} + \|\tilde{u}\|_{H^1(\tilde{\Omega})}), \tag{12}$$

since $\tilde{\Omega}_{-\mathcal{O}}$ is away from the singular points. Then, using the fact that $\vartheta$ is bounded above and below from 0 on $\Omega \setminus \mathcal{O}$ and the relations in Lemma 2.9, we have

$$\begin{aligned}
2\pi |u|^2_{\mathcal{K}^2_{a+1,+}(\Omega\setminus\mathcal{O})} &\le 2\pi \int_{\Omega\setminus\mathcal{O}} \vartheta^{2-2a}\Big((\partial_r^2 u)^2 + (\partial_z^2 u)^2 + 2(\partial_r\partial_z u)^2 + \Big(\frac{\partial_r u}{r}\Big)^2\Big) r dr dz \\
&= \int_0^{2\pi}\int_{\Omega\setminus\mathcal{O}} \vartheta^{2-2a}\Big((\partial_r^2 \tilde{u})^2 + (\partial_z^2 \tilde{u})^2 + 2(\partial_r\partial_z \tilde{u})^2 + \Big(\frac{\partial_r \tilde{u}}{r}\Big)^2\Big) r dr dz d\theta \\
&\le C \int_{\tilde{\Omega}_{-\mathcal{O}}} \Big((\partial_x^2 \tilde{u})^2 + (\partial_y^2 \tilde{u})^2 + (\partial_z^2 \tilde{u})^2 + 2(\partial_x\partial_y \tilde{u})^2 + 2(\partial_x\partial_z \tilde{u})^2 + 2(\partial_y\partial_z \tilde{u})^2\Big) dx dy dz \\
&\le C|\tilde{u}|^2_{H^2(\tilde{\Omega}_{-\mathcal{O}})}.
\end{aligned} \tag{13}$$

Therefore, based on (11)–(13), and Proposition 2.2, we obtain

$$\begin{aligned}
|u|_{\mathcal{K}^2_{a+1,+}(\Omega\setminus\mathcal{O})} &\le C|\tilde{u}|_{H^2(\tilde{\Omega}_{-\mathcal{O}})} \le C(\|\tilde{f}\|_{L^2(\tilde{\Omega}_{-\mathcal{P}})} + \|\tilde{u}\|_{H^1(\tilde{\Omega})}) \\
&\le C(\|f\|_{L^2_r(\Omega\setminus(\cup\mathcal{V}_i/4))} + \|u\|_{H^1_r(\Omega)}) \\
&\le C(\|f\|_{L^2_r(\Omega\setminus(\cup\mathcal{V}_i/4))} + \|u\|_{\mathcal{K}^1_{a+1,r}(\Omega)}) \\
&\le C\|f\|_{\mathcal{K}^0_{a-1,r}(\Omega)} \le C\|f\|_{L^2_r(\Omega)}.
\end{aligned}$$

For the estimates near the vertices, we need to distinguish the vertices away from the $z$-axis and those on the $z$-axis.

For a vertex $Q_i$ that is not on the $z$-axis, in its neighborhood $\mathcal{V}_i/2$, $r$ is bounded below from 0. Therefore, Eq. (4) is elliptic with smooth coefficients and the zero boundary condition. The regularity result is well known in weighted Sobolev spaces equivalent to $\mathcal{K}^2_{a+1,+}(\mathcal{V}_i/2)$. Namely, for $0 \le a < 1$,

$$\begin{aligned}
|u|^2_{\mathcal{K}^2_{a+1,+}(\mathcal{V}_i/2)} &\le C \int_{\mathcal{V}_i/2} \vartheta^{2-2a}\big((\partial_r^2 u)^2 + (\partial_z^2 u)^2 + (\partial_r\partial_z u)^2 + (\partial_r u)^2\big) dr dz \\
&\le C \int_{\mathcal{V}_i} \vartheta^{2-2a} f^2 dr dz \le C\|f\|^2_{\mathcal{K}^0_{a-1,r}(\mathcal{V}_i)} \le C\|f\|^2_{L^2_r(\mathcal{V}_i)}.
\end{aligned}$$

Let $Q_i$ be the origin in the local polar coordinates $(\rho, \phi)$. A simple proof of the estimate above is obtained by using the Mellin transform, a partition of unity of the form $\phi_n(\rho) := \phi(\rho - n)$, and applying the standard regularity results for smooth domains to the function $\phi_n u$. See for example [16,21,39,41] for details.

For a vertex $Q_i$ on the $z$-axis, the desired result on $\mathcal{V}_i/2$ can be derived from the regularity of $\tilde{u}$ as follows (identifying $u$ with $\tilde{u}$ as in Proposition 2.2). Let $\tilde{\Omega}_{\mathcal{V}_i} := \mathcal{V}_i \times [0, 2\pi) \subset \tilde{\Omega}$ (resp. $\tilde{\Omega}_{\mathcal{V}_i/2} := \mathcal{V}_i/2 \times [0, 2\pi) \subset \tilde{\Omega}$). Then, we have the following weighted estimate on $\tilde{\Omega}_{\mathcal{V}_i}$ from [32,33] for the three-dimensional vertex,

$$\int_{\tilde{\Omega}_{\mathcal{V}_i/2}} \vartheta^{2-2a}\big((\partial_x^2 \tilde{u})^2 + (\partial_y^2 \tilde{u})^2 + (\partial_z^2 \tilde{u})^2 + (\partial_x\partial_y \tilde{u})^2 + (\partial_x\partial_z \tilde{u})^2 + (\partial_y\partial_z \tilde{u})^2\big) dx dy dz \le C \int_{\tilde{\Omega}_{\mathcal{V}_i}} \vartheta^{2-2a} \tilde{f}^2 dx dy dz. \tag{14}$$

Note that (14) is in fact the estimate in the weighted space $\mathcal{K}^m_a$ for singular vertices in [32,33], where a similar proof to two-dimensional vertices was carried out, with a partition of unity in a three-dimensional domain. Thus, for $0 \le a < 1$,

$$\begin{aligned}
2\pi |u|^2_{\mathcal{K}^2_{a+1,+}(\mathcal{V}_i/2)} &\le 2\pi \int_{\mathcal{V}_i/2} \vartheta^{2-2a}\Big((\partial_r^2 u)^2 + (\partial_z^2 u)^2 + 2(\partial_r\partial_z u)^2 + \Big(\frac{\partial_r u}{r}\Big)^2\Big) r dr dz \\
&= \int_{\tilde{\Omega}_{\mathcal{V}_i/2}} \vartheta^{2-2a}\big((\partial_x^2 \tilde{u})^2 + (\partial_y^2 \tilde{u})^2 + (\partial_z^2 \tilde{u})^2 + 2(\partial_x\partial_y \tilde{u})^2 + 2(\partial_x\partial_z \tilde{u})^2 + 2(\partial_y\partial_z \tilde{u})^2\big) dx dy dz \\
&\le C \int_{\tilde{\Omega}_{\mathcal{V}_i}} \vartheta^{2-2a} \tilde{f}^2 dx dy dz = 2\pi C \int_{\mathcal{V}_i} \vartheta^{2-2a} f^2 r dr dz \le 2\pi C\|f\|^2_{L^2_r(\mathcal{V}_i)}.
\end{aligned}$$

Adding up all estimates completes the proof of this theorem. $\square$

Note that we can extend the above proof to $f \in \mathcal{K}^0_{a-1,r}$ for $0 \le a < 1$ and obtain $\|u\|_{\mathcal{K}^2_{a+1,+}} \le C\|f\|_{\mathcal{K}^0_{a-1,r}}$.

### 3.3. The Fredholm property

Based on the well-posedness and regularity results established in the previous two subsections, we shall further study the Fredholm property of the operator $\mathcal{L}$ in the weighted Sobolev space $\mathcal{K}^m_{a,r}$. An application of this study on the FEM will be shown in Section 4.

Recall that a *continuous* operator $A : X \to Y$ between Banach spaces is Fredholm if the kernel of $A$ (that is, the space $\ker(A) := \{Ax = 0\}$) and $Y/AX$ are finite dimensional spaces. We also define its index by the formula $\operatorname{ind}(A) = \dim\ker(A) - \dim(Y/AX)$. Then, it is possible to determine $\eta$ in Theorem 3.2 by the Fredholm property of the axisymmetric operator $\mathcal{L}$. Before we proceed with the discussion on this property, we realize the following lemma.

**Lemma 3.4.** *Let $G \subset \Omega$ be an open subset of the domain that is away from the vertices. Then, the space $H^2_+(G)$ is compactly embedded in $H^1_r(G)$.*

**Proof.** This lemma can be justified by the Rellich–Kondrachov Theorem for usual Sobolev spaces and the isomorphisms in Propositions 2.2 and 2.3, between these weighted Sobolev spaces and the usual Sobolev spaces on the corresponding three-dimensional domain. □

Let $Q_i \in \mathcal{S}$ be a vertex of $\Omega$ away from the $z$-axis. In the neighborhood $\mathcal{V}_i$, by freezing the coefficient at $Q_i$, the local behavior of the solution is determined by the principal part $-\partial_r^2 - \partial_z^2$ of the operator $\mathcal{L}$, since $r$ is bounded away from 0. Let $\imath := \sqrt{-1}$. Then, in the polar coordinates $(\rho, \phi)$, $\alpha_i \le \phi \le \beta_i$ on $\mathcal{V}_i$, the operator pencil $P_i(\tau)$ associated to $\mathcal{L}$ in $\mathcal{V}_i$ is defined by

$$-(\partial_r^2 + \partial_z^2)(\rho^{\imath\tau+\epsilon}\zeta(\phi)) = \rho^{\imath\tau+\epsilon-2}P_i(\tau)\zeta(\phi),$$

where $\zeta(\phi)$ is any smooth function with the zero Dirichlet boundary condition for $\phi = \alpha_i$ and $\phi = \beta_i$. Thus, based on the formula $\partial_r^2 + \partial_z^2 = \rho^{-2}((\rho\partial_\rho)^2 + \partial_\phi^2)$ and $(\rho\partial_\rho)^2\rho^{\imath\tau+\epsilon} = \rho^{\imath\tau+\epsilon}(\imath\tau + \epsilon)^2$, we obtain

$$P_i(\tau) = (\tau - \imath\epsilon)^2 - \partial_\phi^2.$$

Let $\theta_i = \beta_i - \alpha_i$ be the interior angle of the corner with vertex $Q_i$. It is well known that the spectrum of the operator $-\partial_\phi^2$, with zero boundary conditions, is

$$\Sigma_i = \left\{\left(\frac{k\pi}{\theta_i}\right)^2\right\}, \quad k = 1, 2, 3, \ldots, \tag{15}$$

and hence $P_i(\tau)$ is invertible for all $\tau \in \mathbb{R}$, given $\epsilon \ne \pm k\pi/\theta_i$.

On the other hand, for a vertex $Q_i$ on the $z$-axis, we characterize the revolution $\tilde{\Omega}_{\mathcal{V}_i} = \mathcal{V}_i \times [0, 2\pi)$ of $\mathcal{V}_i$ by

$$\tilde{\Omega}_{\mathcal{V}_i} = \{(\rho, \omega),\ 0 < \rho < \tilde{l},\ \omega \in \omega_{Q_i}\},$$

in spherical coordinates, where $\omega_{Q_i} \subset S^2$ is the projection of $\tilde{\Omega}_{\mathcal{V}_i}$ on the unit sphere $S^2$. Then, on $\tilde{\Omega}_{\mathcal{V}_i}$, we have the formula $\mathcal{L}u = -\Delta\tilde{u} = -\rho^{-2}((\rho\partial_\rho)^2 + \rho\partial_\rho + \Delta')\tilde{u}$, for the axisymmetric function $\tilde{u}$, where

$$\Delta' = (\cot\phi)\partial_\phi + \partial_\phi^2 + (\sin^2\phi)^{-2}\partial_\theta^2$$

denotes the Laplace–Beltrami operator on $\omega_{Q_i}$. The operator pencil for $-\Delta$ on $\tilde{\Omega}_{\mathcal{V}_i}$ is thus given by

$$P_i(\tau) = -((\imath\tau + \epsilon - 1/2)(\imath\tau + \epsilon + 1/2) + \Delta').$$

Inheriting the boundary condition from the original equation (2), that is, the zero boundary condition, and taking $\partial_\theta\tilde{u} = 0$ into account, the smallest real eigenvalue of the operator $-\Delta'$ on $\omega_{Q_i}$ is strictly positive $\lambda_{i,1} > 0$, (see [11]), and can be computed numerically. Therefore, $P_i(\tau)$ is invertible for all $\tau \in \mathbb{R}$, when $|\epsilon| < \sqrt{\lambda_{i,1} + 1/4}$.

Recall the isometries between different spaces from Proposition 2.2. Note that $\mathcal{K}^m_{a,r}(G)$ is equivalent to $H^m_r(G)$ for $G \subset \Omega$ away from the vertices, and hence $\mathcal{K}^2_{a,+}(G)$ is compactly embedded in $\mathcal{K}^1_{a,r}(G)$ by Lemma 3.4. Define

$$\eta_1 := \min_i(\sqrt{\lambda_{i,1} + 1/4}).$$

Thus, for $|\epsilon| < \eta_1$ and $\epsilon \ne \pm k\pi/\theta_i$, following Kondratiev's method [21], we obtain the Fredholm conditions on the operator $\vartheta^{-\epsilon}\mathcal{L}\vartheta^{\epsilon} : \mathcal{K}^2_{1,+} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^0_{-1,r}$, which implies that $\mathcal{L} : \mathcal{K}^2_{\epsilon+1,+} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^0_{\epsilon-1,r}$ is Fredholm by Lemma 2.11. See [21,39] and references therein for more details on the Kondratiev's method.

For $\epsilon$ out of the range above, the operator $\mathcal{L}$ may not be Fredholm, or is Fredholm but has a non-zero index, and hence is not invertible. For the computation of non-zero indices of Fredholm operators, we refer to [42,43]. Now, we are in the position to specify the index $a$ and improve our well-posedness result.

**Theorem 3.5.** *Define* $\eta := \min(1, \pi/\theta_i, \sqrt{\lambda_{i,1} + 1/4})$. *Then, for any* $0 \le a < \eta$, *there is a unique solution* $u \in \mathcal{K}^2_{a+1,+} \cap \{v|_{\Gamma_0} = 0\}$ *for Eq.* (4), *provided that* $f \in \mathcal{K}^0_{a-1,r}$.

**Proof.** Theorems 3.2 and 3.3, Lemma 2.16 and the discussion above show that, for $a = 0$, the operator $\mathcal{L} : \mathcal{K}^2_{a+1,+} \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}^0_{a-1,r}$ is Fredholm with index zero, since it is invertible. By the homotopy invariance of the index, $\mathcal{L}$ is Fredholm with index zero for $0 \le a < \eta$. Note that the kernel of $\mathcal{L}$ is non-increasing as $a$ increases. Therefore, $\mathcal{L}$ is injective between these spaces. Since the index is zero, we conclude it is in fact a bijection for $0 \le a < \eta$. □

## 4. The finite element estimates in weighted spaces

In this section, we analyze the finite element approximation for the solution of Eq. (4), especially for singular solutions, in weighted Sobolev spaces. Precisely, let $\mathcal{T} := \{T_i\}$ be a triangulation of $\Omega$ with triangles $T_i$. Denote by

$$S := S(\mathcal{T}, 1) \subset H^1_r \cap \{v|_{\Gamma_0} = 0\} = \mathcal{K}^1_{1,r} \cap \{v|_{\Gamma_0} = 0\}$$

the finite element space associated to the linear Lagrange triangle. Then, the finite element solution $u_S \in S$ is defined by

$$a(u_S, v_S) = \int_\Omega (\partial_r u_S \partial_r v_S + \partial_z u_S \partial_z v_S) r dr dz = \int_\Omega f v_S r dr dz, \tag{16}$$

for any $v_S \in S$. To obtain an error estimate, we shall first establish an approximation result assuming the solution is sufficient regular in $H^2_r$. Then we describe a simple and explicit construction of a sequence of triangulations $\mathcal{T}_n$, suitably graded to points where singularities in the solution occur, such that the following *quasi-optimal* rate of convergence can be achieved

$$\|u - u_n\|_{H^1_r} \le C \dim(S_n)^{-1/2} \|f\|_{L^2_r}, \quad \forall f \in L^2_r,$$

where $S_n = S(\mathcal{T}_n, 1)$ is the finite element space on the mesh $\mathcal{T}_n$ and $u_n := u_{S_n} \in S_n$ is the finite element solution.

We first need the following estimate from Céa's Lemma for further analysis.

**Lemma 4.1.** *Given the finite element solution* $u_S$ *defined above, then there exists a constant* $C > 0$, *independent of* $u$, *such that*

$$\|u - u_S\|_{\mathcal{K}^1_{1,r}} \le C \inf_{\chi \in S} \|u - \chi\|_{\mathcal{K}^1_{1,r}}.$$

**Proof.** The proof is standard. Let $\|u\|^2_a := a(u, u)$. Indeed, we have

$$\|u - u_S\|_a = \inf_{\chi \in S} \|u - \chi\|_a,$$

because $u_S$ is the projection of $u$ onto $S$ in the $a$-inner product. The result then follows from Céa's Lemma and the equivalence of the $a$-norm and the $\mathcal{K}^1_{1,r}$-norm, given the Dirichlet boundary condition on $\Gamma_0$ (Lemma 3.1). □

### 4.1. Approximation in the space $H^m_r$

In the rest of the paper, we require that all triangles of the triangulation $\mathcal{T}$ are shape-regular, and adjacent triangles have comparable size. Namely, let $T_i, T_j \in \mathcal{T}$ be two triangles, such that $\bar{T}_i \cap \bar{T}_j \ne \emptyset$, then there exists a constant $C_0$,

$$\max_{T_i, T_j \in \mathcal{T}_n} \frac{\operatorname{diam} T_i}{\operatorname{diam} T_j} \le C_0. \tag{17}$$

The Lagrange interpolation operator $\mathcal{I} : \mathcal{C}^0 \to S$ is such that for any $v \in \mathcal{C}^0(\bar{\Omega})$, $\mathcal{I}v(\mathbf{x}_i) = v(\mathbf{x}_i)$ at the nodes $\mathbf{x}_i$ of each triangle. In addition, for a sub-domain $G \subset \Omega$, we denote by $P_k(G)$ the set of polynomials of degree $\le k$ on $G$. In this section, the constant $C > 0$ in our estimates will in general depend on the shape regularity of the triangles in $\mathcal{T}$, but not on the solution $u$ or the given data $f$.

We first state a lemma from [3], regarding the polynomial approximation property in the weighted Sobolev space. It is an extension of the well-known approximation result in the usual Sobolev space.

**Lemma 4.2.** *For a compact set* $K \subset \bar{\Omega}$, *let* $h_K = \operatorname{diam} K < 1$ *be its diameter. Suppose* $K$ *is star-shaped with respect to a ball of radius* $\delta h_K$. *If* $K \cap \{r = 0\} \ne \emptyset$,

$$\inf_{p \in P_1(K)} (h_K^{-1} \|v - p\|_{L^2_r(K)} + |v - p|_{H^1_r(K)}) \le C h_K |v|_{H^2_r(K)}, \quad \forall v \in H^2_r(K),$$

*where the constant* $C$ *depends on* $\delta$, *but not on* $v$ *or* $h_K$.

**Proof.** Since the proof is rather long and similar to the process in [44,45] for usual Sobolev spaces, we only give a sketch. One can construct a linear function $p$ by using Taylor's Theorem for smooth functions. Then, based on the estimates on the weight $r$ in the space $H^m_r$ and the estimates on the residue $|v - p|$, the desired result can be obtained for $v \in H^2_r(K)$ using the density argument. The complete proof of this lemma can be found in the long version of this paper [46]. See also the proof in [3] for the upper bound of $\inf_{p \in P_1(K)} \|v - p\|_{L^2_r(K)}$. □

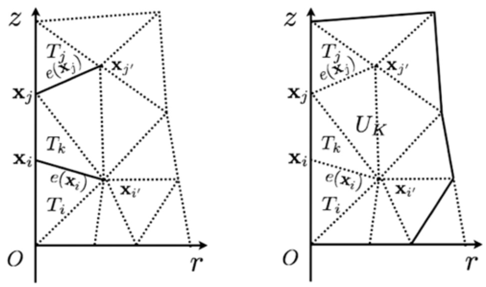


**Fig. 2.** A selection of $(e(\mathbf{x}_i), T_i)$ and $(e(\mathbf{x}_j), T_j)$ associated to $T_k$ (left); the patch $U_k$ based on this selection (right).

**Remark 4.3.** A weighted Sobolev embedding result $\|v\|_{L^\infty(\bar{\Omega})} \le C\|v\|_{H^2_r(\Omega)}$ is obtained in [8]. Thus, using the approximation property (Lemma 4.2) and the nodal interpolation operator, it is possible to analyze the convergence rate for the finite element solution, provided the real solution $u \in H^2_r(\Omega)$. We here, however, present a different approach by introducing a new interpolation operator $\Pi : H^2_r(\Omega) \to S(\mathcal{T}, 1)$ based on a local regularization process (Definition 4.4). Exploiting critical properties of functions in the weighted space for the finite element analysis, this technique shall allow us to get sharp error analysis in this paper for singular solutions, as well as provide useful tools in the future work for more complex axisymmetric problems with low-regularity data (e.g., [37]).

**Definition 4.4.** For each node $\mathbf{x}_i$ on the $z$-axis, we associate $\mathbf{x}_i$ with an edge $e(\mathbf{x}_i)$ and a triangle $T_i$, such that $\mathbf{x}_i$ is an endpoint of $e(\mathbf{x}_i)$, $e(\mathbf{x}_i)$ does not lie on the $z$-axis, and $e(\mathbf{x}_i)$ is an edge of $T_i$. Denote by $\mathbf{x}_{i'}$ the other endpoint of $e(\mathbf{x}_i)$ (see Fig. 2). In addition, if $\mathbf{x}_i$ is the endpoint of $\Gamma_0 \subset \partial\Omega$, we require the associated edge $e(\mathbf{x}_i)$ lies on $\Gamma_0$ to preserve the boundary condition. Define the operator $\pi_i : H^2_r(T_i) \to \mathbb{R}$

$$\pi_i v = \frac{\int_{e(\mathbf{x}_i)} (\mathbf{t} \cdot \nabla v) r ds}{\int_{e(\mathbf{x}_i)} r ds} |e(\mathbf{x}_i)|,$$

where $\mathbf{t}$ denotes the unit vector parallel to $e(\mathbf{x}_i)$, pointing from $\mathbf{x}_{i'}$ to $\mathbf{x}_i$, and $\nabla = (\partial_r, \partial_z)$. We then define the new interpolation operator $\Pi : H^2_r(\Omega) \to S(\mathcal{T}, 1)$,

$$\Pi v := \sum_{i, \mathbf{x}_i \notin \{r=0\}} v(\mathbf{x}_i)\psi_i + \sum_{i, \mathbf{x}_i \in \{r=0\}} (v(\mathbf{x}_{i'}) + \pi_i v)\psi_i,$$

where $\psi_i \in S(\mathcal{T}, 1)$ is the usual linear basis function associated with $\mathbf{x}_i$.

Note that the associations between $\mathbf{x}_i$ and $e(\mathbf{x}_i)$ and between $e(\mathbf{x}_i)$ and $T_i$, are not unique. One can select any edge connected to $\mathbf{x}_i$ as $e(\mathbf{x}_i)$ and any triangle including $e(\mathbf{x}_i)$ as $T_i$, as long as they satisfy the conditions in Definition 4.4. It is also clear that $\Pi v_n = v_n$ for any $v_n \in S(\mathcal{T}, 1)$. We then have the following approximation property of $\Pi v$ away from the $z$-axis.

**Lemma 4.5.** *Let $G \subset \Omega$ be a sub-domain such that $r \ge Mh$ on $G$, for $0 < h < 1$. Let $\mathcal{T} = \{T_i\}$ be the triangulation of $G$ with quasi-uniform triangles of size $h$. Then,*

$$\|v - \Pi v\|_{H^1_r(G)} \le Ch|v|_{H^2_r(G)}, \quad \forall v \in H^2_r(G),$$

*where the constant $C$ depends on $G$ and the shape regularity of the triangles.*

**Proof.** Note that on $G$, $\Pi v = \mathcal{I} v$. Therefore, by the usual estimate in Sobolev spaces, for any triangle $T_i \in \mathcal{T}$, we obtain

$$\|v - \Pi v\|_{H^1(T_i)} = \|v - \mathcal{I} v\|_{H^1(T_i)} \le Ch|v|_{H^2(T_i)}.$$

Let $r_{i,min}$ and $r_{i,max}$ be the smallest and the largest distance from any point in $\bar{T}_i$ to the $z$-axis, respectively. Then, there exists a constant $M_1$, such that $1 < \max_i(r_{i,max}/r_{i,min}) \le M_1$, since $r \ge Mh$ on $G$. Therefore,

$$\begin{aligned}\|v - \Pi v\|_{H^1_r(T_i)} &\le r_{i,max}^{1/2}\|v - \Pi v\|_{H^1(T_i)} \le Cr_{i,max}^{1/2} h|v|_{H^2(T_i)} \\ &\le CM_1 r_{i,min}^{1/2} h|v|_{H^2(T_i)} \le CM_1 h|v|_{H^2_r(T_i)}.\end{aligned}$$

The proof is thus completed by adding up the estimates for all triangles. □

We now define some special terms that we will use often in the text below. By an *a-node*, we mean a node of the triangulation that does not lie on the $z$-axis; by a *z-node*, we mean a node on the $z$-axis.

Recall the associated edge $e(\mathbf{x}_i)$ to each $z$-node $\mathbf{x}_i$ from Definition 4.4. For any triangle $T_k \in \mathcal{T}$ whose closure intersects the $z$-axis, let $\mathcal{Z}_k = \{\mathbf{x}_i\}$ be the union of its $z$-nodes. We associate to $T_k$ the following patch

$$U_k := \text{interior}(\cup_{\mathbf{x}_i \in \mathcal{Z}_k}\{\bar{T}_j,\ \bar{T}_j \cap \bar{e}(\mathbf{x}_i) \neq \emptyset\} \cup \{\bar{T}_j,\ \bar{T}_j \cap \bar{T}_k \neq \emptyset\}), \quad \forall T_j \in \mathcal{T}. \tag{18}$$

Namely, the open set $U_k \subset \Omega$ forms a neighborhood of $\cup_{\mathbf{x}_i \in \mathcal{Z}_k}\{\bar{e}(\mathbf{x}_i)\} \cup \bar{T}_k$. Therefore, by (17), $U_k$ is the union of finite overlapped domains $D_i$. Each $D_i$ is star-shaped with respect to a ball of radius $\geq C_1 h_k = C_1 \text{diam}\, T_k$, for $C_1$ depending on the shape regularity of the triangles. For example, every $D_i$ can be the union of two triangles in $U_k$, sharing a common edge (see Fig. 2). Then, based on Lemma 4.2 and the standard approximation theory in usual Sobolev spaces [44,47], on every $D_i$, we have $\inf_{p \in P_1(D_i)}(h_k^{-1}\|v-p\|_{L_r^2(D_i)} + |v-p|_{H_r^1(D_i)}) \leq Ch_k|v|_{H_r^2(D_i)}$. Hence, by a theorem developed in [45],

$$\inf_{p \in P_1(U_k)}(h_k^{-1}\|v-p\|_{L_r^2(U_k)} + |v-p|_{H_r^1(U_k)}) \leq Ch_k|v|_{H_r^2(U_k)} \tag{19}$$

for $C$ depending on the domains $D_i$ and the triangulation.

Thus, we have the following estimates in the neighborhood of the $z$-axis.

**Lemma 4.6.** *For any $T_k \in \mathcal{T}$, with $\bar{T}_k \cap \{r=0\} \neq \emptyset$, let $h_k = \text{diam}\, T_k < 1$. Then,*

$$\|v - \Pi v\|_{H_r^1(T_k)} \leq Ch_k|v|_{H_r^2(U_k)}, \quad \forall v \in H_r^2(U_k),$$

*where $U_k$ represents the patch defined in (18) and the constant $C$ depends on the shape regularity of the triangulation.*

**Proof.** Denote by $\psi_j$ the linear basis function associated to the node $\mathbf{x}_j$. Then, for any $\psi_j$ whose support intersects $T_k$, noting $\max(r(\bar{T}_k))$ is comparable with $h_k$, we first derive the following estimate,

$$\begin{aligned}\|\psi_j\|_{H_r^1(T_k)} &= \left(\int_{T_k}\psi_j^2 rdrdz + \int_{T_k}(|\partial_r\psi_j|^2 + |\partial_z\psi_j|^2)rdrdz\right)^{1/2} \\ &\leq C\left(\int_{T_k}(1+h_k^{-2})rdrdz\right)^{1/2} \leq C(h_k^{-2}h_k^2h_k)^{1/2} = Ch_k^{1/2},\end{aligned}$$

where $C$ depends on the triangulation. Note for any $p \in P_1(T_k)$,

$$\|v - \Pi v\|_{H_r^1(T_k)} \leq \|v-p\|_{H_r^1(T_k)} + \|\Pi(v-p)\|_{H_r^1(T_k)}.$$

Setting $w = v - p$, we shall verify the estimate for $\|w\|_{H_r^1(T_k)}$ and $\|\Pi w\|_{H_r^1(T_k)}$.

Recall that $\mathcal{Z}_k$ is the union of the $z$-nodes of $T_k$ and for any node $\mathbf{x}_i \in \mathcal{Z}_k$, there is an associated edge $e(\mathbf{x}_i)$. The other endpoint $\mathbf{x}_{i'}$ of $e(\mathbf{x}_i)$ is an $a$-node (Definition 4.4). Meanwhile, we define the union of $a$-nodes associated to $T_k$,

$$\mathcal{A}_k = \{\mathbf{x}_{i'},\ \mathbf{x}_i \in \mathcal{Z}_k\} \cup \{a\text{-nodes} \in \bar{T}_k\}.$$

Note that, for any $\mathbf{x}_l \in \mathcal{A}_k$, we can associate it to a triangle $T_l \in U_k$, away from the $z$-axis, such that $\mathbf{x}_l \in T_l$ is one of its vertices. Denote by $\hat{T}$ the standard reference triangle with diam $\hat{T} = 1$. Then, the affine mapping $F$ between $T_l$ and $\hat{T}$ is defined by $F(T_l) = \hat{T}$ and $w(\mathbf{x}) = \hat{w}(\hat{\mathbf{x}}) = \hat{w}(F(\mathbf{x}))$. Therefore, by the usual Sobolev embedding Theorem, a scaling argument, and the definition of the norms,

$$\begin{aligned}|w(\mathbf{x}_l)| &\leq |\hat{w}|_{L^\infty(\hat{T})} \leq C\|\hat{w}\|_{H^2(\hat{T})} \leq C(h_k^{-1}\|w\|_{L^2(T_l)} + |w|_{H^1(T_l)} + h_k|w|_{H^2(T_l)}) \\ &\leq C(h_k^{-3/2}\|w\|_{L_r^2(T_l)} + h_k^{-1/2}|w|_{H_r^1(T_l)} + h_k^{1/2}|w|_{H_r^2(T_l)}).\end{aligned}$$

In the last inequality, we used the fact that the ratio $\max(r(\bar{T}_l))/\min(r(\bar{T}_l)) \leq M_1$ and $\min(r(\bar{T}_l))$ is comparable with $h_k$, since $\bar{T}_l$ does not intersect the $z$-axis.

Therefore, by Definition 4.4 and the estimates above, we have

$$\begin{aligned}\|\Pi w\|_{H_r^1(T_k)} &\leq \sum_{i,\mathbf{x}_i \in \mathcal{Z}_k}|w(\mathbf{x}_{i'}) + \pi_i w|\|\psi_i\|_{H_r^1(T_k)} + \sum_{l,\mathbf{x}_l \in \mathcal{A}_k \setminus \{\mathbf{x}_{i'}\}}|w(\mathbf{x}_l)|\|\psi_l\|_{H_r^1(T_k)} \\ &\leq C(h_k^{1/2}\sum_{i,\mathbf{x}_i \in \mathcal{Z}_k}|\pi_i w| + \sum_{l,\mathbf{x}_l \in \mathcal{A}_k}(h_k^{-1}\|w\|_{L_r^2(T_l)} + |w|_{H_r^1(T_l)} + h_k|w|_{H_r^2(T_l)})).\end{aligned} \tag{20}$$

Furthermore, by the shape regularity of the triangulation and (17), we notice that $Ah_k^2 \leq \int_{e(\mathbf{x}_i)} rds \leq Bh_k^2$ and $Ah_k \leq |e(\mathbf{x}_i)| \leq Bh_k$, for $A, B > 0$ depending on the triangulation. We then focus on the estimate for $|\pi_i w|$.

Let $T_i \subset U_k$ be a triangle with $e(\mathbf{x}_i)$ as an edge.

Case 1: $\bar{T}_i \cap \{r = 0\}$ is a single node $\mathbf{x}_i$. By Definition 4.4, the trace estimate in Proposition 2.6, and the Hölder inequality, we obtain

$$\begin{aligned}
|\pi_i w| &= \left| \frac{\int_{e(\mathbf{x}_i)} (\mathbf{t}\cdot\nabla w) r ds}{\int_{e(\mathbf{x}_i)} r ds} e(\mathbf{x}_i) \right| \le C h_k^{-1} \left| \int_{e(\mathbf{x}_i)} (\mathbf{t}\cdot\nabla w) r ds \right| \\
&\le C h_k^{-1} \left| \left| \int_{e(\mathbf{x}_i)} |\nabla w|^2 r^2 ds \right|^{1/2} \left| \int_{e(\mathbf{x}_i)} ds \right| \right|^{1/2} \\
&\le C h_k^{1/2} (h_k^{-2} |w|^2_{H^1_r(T_i)} + |w|^2_{H^2_r(T_i)})^{1/2} \le C h_k^{1/2} (h_k^{-1} |w|_{H^1_r(T_i)} + |w|_{H^2_r(T_i)}).
\end{aligned}$$

Case 2: $\bar{T}_i \cap \{r = 0\}$ is a line segment. Similarly, by Proposition 2.6, we have

$$\begin{aligned}
|\pi_i w| &\le C h_k^{-1} \left| \int_{e(\mathbf{x}_i)} (\mathbf{t}\cdot\nabla w) r ds \right| \le C h_k^{-1} \left| \int_{e(\mathbf{x}_i)} |\nabla w|^2 r ds \right|^{1/2} \left| \int_{e(\mathbf{x}_i)} r ds \right|^{1/2} \\
&\le C h_k^{1/2} (h_k^{-2} |w|^2_{H^1_r(T_i)} + |w|^2_{H^2_r(T_i)})^{1/2} \le C h_k^{1/2} (h_k^{-1} |w|_{H^1_r(T_i)} + |w|_{H^2_r(T_i)}).
\end{aligned}$$

Hence, in both cases, combining (20), the estimates yield

$$\|\Pi w\|_{H^1_r(T_k)} \le C\Big( \sum_{i,\mathbf{x}_i\in\mathcal{Z}_k} (|w|_{H^1_r(T_i)} + h_k |w|_{H^2_r(T_i)}) + \sum_{l,\mathbf{x}_l\in\mathcal{A}_k} (h_k^{-1}\|w\|_{L^2_r(T_l)} + |w|_{H^1_r(T_l)} + h_k |w|_{H^2_r(T_l)}) \Big).$$

Combining the estimates above and (19), we obtain for any $p \in P_1(U_k)$,

$$\begin{aligned}
&\|v - \Pi v\|_{H^1_r(T_k)} \le \|v - p\|_{H^1_r(T_k)} + \|\Pi(v - p)\|_{H^1_r(T_k)} \\
&\quad \le C(h_k^{-1}\|v - p\|_{L^2_r(U_k)} + |v - p|_{H^1_r(U_k)} + h_k |v - p|_{H^2_r(U_k)}) \le C h_k |v|_{H^2_r(U_k)}. \quad \square
\end{aligned}$$

**Remark 4.7.** It is clear that summing up the estimates in Lemma 4.6 for all triangles near the $z$-axis and combining with Lemma 4.5, one has $\|v - \Pi v\|_{H^1_r} \le Ch|v|_{H^2_r}$, if the triangulation $\mathcal{T}$ contains quasi-uniform triangles of size $h$. Let $u_h$ be the finite element solution of Eq. (4). Thus, based on Céa's Lemma, we have $\|u - u_h\|_{H^1_r} \le C\|u - \Pi u\|_{H^1_r} \le Ch|u|_{H^2_r}$, provided that $u \in H^2_+ \subset H^2_r$ on these meshes.

**Remark 4.8.** Note that in order to obtain an estimate as in Lemma 4.6, it is not necessary to include every triangle in the neighborhood of $\cup_{\mathbf{x}_i\in\mathcal{Z}_k}\{\bar{e}(\mathbf{x}_i)\} \cup \bar{T}_k$ (patch $U_k$). Based on the analysis above, the error estimate still holds on $U_k$, as long as $U_k$ satisfies the following criteria.

1. Besides $T_k$, $U_k$ should contain every triangle $T_i$ associated to $e(\mathbf{x}_i)$, $\forall \mathbf{x}_i \in \mathcal{Z}_k$, and every triangle $T_l$ associated to $\mathbf{x}_l$, $\forall \mathbf{x}_l \in \mathcal{A}_k$.
2. $U_k$ is the union of finite overlapped domains $D_i$, each of which is star-shaped with respect to a ball of radius $\ge Ch_k$, such that the estimate (19) holds by [48].

This allows us to simplify our presentation in the subsections below, by modifying the definition of $U_k$ while keeping the criteria above.

### 4.2. Approximation in the space $\mathcal{K}^m_{a,r}$

The approximation results in Remark 4.7 provide the analogy of the best polynomial approximation in the usual Sobolev space, when the solution is regular enough (in $H^2_+$). However, it is very possible that the solution of Eq. (4) possesses singularities in $H^2_+$ near the vertices of the domain, which will destroy the optimal convergence rate. From now on, we shall extend these approximation results to the space $\mathcal{K}^m_{a,r}$ for possible singularities and describe a simple and explicit construction of a sequence of finite element spaces, such that the *quasi-optimal* convergence rate can be achieved for singular solutions.

Recall the operator $\Pi : H^2_r \to S(\mathcal{T}, 1)$. For a function $v \in \mathcal{K}^2_{a,r} \cap \{v|_{\Gamma_0} = 0\}$, $a \ge 1$, we change its definition on the vertex set of the domain. We define

$$(\Pi v)(Q_i) = 0 \qquad \forall Q_i \in \mathcal{S}, \tag{21}$$

and let $\Pi v$ remain the same on the other nodes as in Definition 4.4, since $v \in H^2_r(G)$ for any $G \subset \Omega$ away from the vertices. Recall the open set $\Gamma_1 \subset \partial\Omega$. Then, for any edge $e(\mathbf{x}_i)$ associated to a node $\mathbf{x}_i \in \Gamma_1$ (Definition 4.4), we require that the assigned triangle $\bar{T}_i$ does not contain any point from the vertex set $\mathcal{S}$. A possible selection of $T_i$ is given in Remark 4.10.

We first study the local behavior with respect to dilations of a function $v \in \mathcal{K}^m_{a,r}$, in the neighborhood $\mathcal{V}_i$ of $Q_i \in \mathcal{S}$. We consider a new coordinate system that is a simple translation of the old $rz$-coordinate system, now with $Q_i$ at the origin of the new coordinate system. Let $G_\lambda \subset \mathcal{V}_i$ be a subset, such that $\vartheta \le \xi \le \tilde{l}$ on $G_\lambda$. For $0 < \lambda < 1$, We let $G := \lambda G_\lambda$. Then, we


**Fig. 3.** One $\kappa$-*refinement* for a triangle with a vertex $Q_i \in \mathcal{S}$.

define the dilation of a function $v$ on $G_\lambda$ in the new coordinate system as follows

$$v_\lambda(r,z) := v(\lambda r, \lambda z),$$

for all $(r,z) \in G_\lambda \subset \mathcal{V}_i$. (This definition makes sense, since $Q_i$ is the origin in the new coordinate system.) We shall need the following dilation lemma.

**Lemma 4.9.** *For $0 < \lambda < 1$, let $G_\lambda \subset \mathcal{V}_i$ be an open subset, and $G := \lambda G_\lambda \subset \mathcal{V}_i$. Then, if $Q_i \in \{r = 0\}$, $\|v_\lambda\|_{\mathcal{K}^m_{a,r}(G_\lambda)} = \lambda^{a-3/2}\|v\|_{\mathcal{K}^m_{a,r}(G)}$; if $Q_i \notin \{r = 0\}$, $C_1\lambda^{a-1}\|v\|_{\mathcal{K}^m_{a,r}(G)} \le \|v_\lambda\|_{\mathcal{K}^m_{a,r}(G_\lambda)} \le C_2\lambda^{a-1}\|v\|_{\mathcal{K}^m_{a,r}(G)}$, for constants $C_1, C_2 > 0$ depending on $\Omega$, $\forall v \in \mathcal{K}^m_{a,r}(\mathcal{V}_i)$, $m = 0, 1, 2$.*

**Proof.** The proof is based on the change of variables $s = \lambda r$, $t = \lambda z$. Note that on both $G_\lambda \subset \mathcal{V}_i$ and $G \subset \mathcal{V}_i$, $\vartheta(r,z)$ is equal to the distance from $(r,z)$ to $Q_i$, therefore $\vartheta(r,z) = \lambda^{-1}\vartheta(s,t)$. Then, if $Q_i \in \{r = 0\}$,

$$\begin{aligned}
\|v_\lambda(r,z)\|^2_{\mathcal{K}^m_{a,r}(G_\lambda)} &= \sum_{j+k\le m}\int_{G_\lambda} |\vartheta^{j+k-a}(r,z)\partial_r^j\partial_z^k v_\lambda(r,z)|^2 r drdz \\
&= \sum_{j+k\le m}\int_G |\lambda^{a-j-k}\vartheta^{j+k-a}(s,t)\lambda^{j+k}\partial_s^j\partial_t^k v(s,t)|^2\lambda^{-3} sdsdt \\
&= \lambda^{2a-3}\sum_{j+k\le m}\int_G |\vartheta^{j+k-a}(s,t)\partial_s^j\partial_t^k v(s,t)|^2 sdsdt = \lambda^{2a-3}\|v\|^2_{\mathcal{K}^m_{a,r}(G)}.
\end{aligned}$$

On the other hand, if $Q_i \notin \{r = 0\}$, we notice $A \le r^{-1} \le B$ on $\mathcal{V}_i$, for constants $A$ and $B$ depending on the domain $\Omega$. Therefore, we have,

$$A\|v(r,z)\|^2_{\mathcal{K}^m_{a,r}(D)} \le \sum_{j+k\le m}\int_D |\vartheta^{j+k-a}(r,z)\partial_r^j\partial_z^k v(r,z)|^2 drdz \le B\|v(r,z)\|^2_{\mathcal{K}^m_{a,r}(D)},$$

where $D \subset \mathcal{V}_i$ is any subset of $\mathcal{V}_i$. Applying the new coordinate system with $Q_i$ at the origin as above, we thus have

$$\begin{aligned}
\|v_\lambda(r,z)\|^2_{\mathcal{K}^m_{a,r}(G_\lambda)} &\le A^{-1}\sum_{j+k\le m}\int_{G_\lambda} |\vartheta^{j+k-a}(r,z)\partial_r^j\partial_z^k v_\lambda(r,z)|^2 drdz \\
&= A^{-1}\sum_{j+k\le m}\int_G |\lambda^{a-j-k}\vartheta^{j+k-a}(s,t)\lambda^{j+k}\partial_s^j\partial_t^k v(s,t)|^2\lambda^{-2} dsdt \\
&= A^{-1}\lambda^{2a-2}\sum_{j+k\le m}\int_G |\vartheta^{j+k-a}(s,t)\partial_s^j\partial_t^k v(s,t)|^2 dsdt \le A^{-1}B\lambda^{2a-2}\|v\|^2_{\mathcal{K}^m_{a,r}(G)}.
\end{aligned}$$

We note that the inequality in the opposite direction can be justified with the same process, which completes the proof. □

For $\mathcal{V}_i := \Omega \cap B(Q_i, \tilde{l})$, let $T_\xi \subset \mathcal{V}_i$ be a triangle with the biggest edge of length $= \xi$ and $Q_i$ is a vertex of $T_\xi$. Denote by $T_{\kappa\xi} \subset T_\xi$ the sub-triangle of $T_\xi$ that has $Q_i$ as a vertex and has all sides parallel to the sides of $T_\xi$. Therefore, $T_{\kappa\xi}$ is similar with $T_\xi$ with the ratio of similarity $\kappa$, $0 < \kappa < 1$. Then, $T_\xi$ is divided into the small triangle $T_{\kappa\xi}$ that has the common vertex $Q_i$ with $T_\xi$ and the trapezoid between the two parallel edges (Fig. 3).

Let $G := T_\xi \setminus T_{\kappa\xi} \subset \mathcal{V}_i$ be the trapezoid. Recall that the triangulation $\mathcal{T}$ of $\Omega$ contains shape-regular triangles and satisfies (17). Suppose all the triangles $T_i \in \mathcal{T}$, satisfying $T_i \cap G \neq \emptyset$, form a *quasi-uniform* triangulation $\mathcal{T}_G$ of $G$.

**Remark 4.10.** In the case $\bar{G} \cap \{r = 0\} \neq \emptyset$, let $\mathcal{Z}_G = \{T_k\}$ be the union of triangles $T_k \in \mathcal{T}_G$, such that $\bar{T}_k \cap \{r = 0\} \neq \emptyset$. To simplify our presentation, for every $T_k \in \mathcal{Z}_G$ that contains an endpoint of the segment $\bar{G} \cap \{r = 0\}$, we define a new patch $U_k^N$ with the same criteria as the patch $U_k$ in Remark 4.8 as follows. For each endpoint $\mathbf{x}_i$ of the segment $\bar{G} \cap \{r = 0\}$, we assign the associated edge $e(\mathbf{x}_i)$ to be on one of the parallel edges of the trapezoid $G$ accordingly. Thus, for each $z$-node $\mathbf{x}_i$ of $\mathcal{T}_G$ and its associated edge $e(\mathbf{x}_i)$, we are able to assign a triangle $T_i \in \mathcal{T}_G$ to $e(\mathbf{x}_i)$, such that $\bar{T}_i$ contains the edge $e(\mathbf{x}_i)$ as in Definition 4.4. From the description, it is clear that $T_i \in \mathcal{Z}_G$ and is away from the vertex $Q_i$. In addition, we assume that for each $a$-node $\mathbf{x}_l$ of $T_k \in \mathcal{Z}_G$, there exists at least one triangle $T_l \in \mathcal{T}_G$ with $\mathbf{x}_l$ as a vertex and $\bar{T}_l \cap \{r = 0\} = \emptyset$. Then, we define the new patch for every triangle $T_k \in \mathcal{Z}_G$, $U_k^N := U_k \cap G$, for $U_k$ from (18). Note that every $U_k^N \subset G$ includes all the triangles needed in the proof of Lemma 4.6 and is the union of finite overlapped domains. Hence, by Remark 4.8, the estimates in Lemma 4.6 still hold if we replace $U_k$ by $U_k^N$.

We then have the following estimate near $Q_i$.

**Lemma 4.11.** *For $0 < \kappa < 1$, let $G = T_\xi \setminus T_{\kappa\xi}$, $Q_i$, $\mathcal{V}_i$, $\mathcal{T}_G$, and $U_k^N$ be as defined above. Let $h$ be the mesh size of $\mathcal{T}_G$. Then,*

$$\|v - \Pi v\|_{\mathcal{K}^1_{1,r}(G)} \leq C(\kappa)\xi^a (h/\xi)\|v\|_{\mathcal{K}^2_{a+1,r}(G)},$$

*for all $v \in \mathcal{K}^2_{a+1,r}(\mathcal{V}_i)$, $a \geq 0$, with $C(\kappa)$ independent of $\xi$, $h$, and $v$.*

**Proof.** Recall the new coordinate system with $Q_i$ as the origin. Let $G_\lambda = \lambda^{-1}G$. Recall the dilation function $v_\lambda(r, z) = v(\lambda r, \lambda z)$. Note that by the definition of $\Pi v$, $(\Pi v)_\lambda = \Pi(v_\lambda)$ on $G_\lambda$. Then, we choose $\lambda = \xi/\tilde{l}$, such that $G_\lambda \subset \mathcal{V}_i$.

For a vertex $Q_i \in \{r = 0\}$, if $\bar{G}_\lambda \cap \{r = 0\} \neq \emptyset$, note that $(U_k^N)_\lambda \subset G_\lambda$. On the other hand, if $\bar{G}_\lambda \cap \{r = 0\} = \emptyset$, by the property (17) of $\mathcal{T}$, the distance to the $z$-axis from $G_\lambda$, $r(G_\lambda) \geq Ch/\lambda$. We thus apply Lemmas 4.5 and 4.6 accordingly to the region $G_\lambda$, based on its relation with the $z$-axis,

$$\begin{aligned}
\|v - \Pi v\|_{\mathcal{K}^1_{1,r}(G)} &= \lambda^{1/2}\|v_\lambda - (\Pi v)_\lambda\|_{\mathcal{K}^1_{1,r}(G_\lambda)} \\
&= \lambda^{1/2}\|v_\lambda - \Pi(v_\lambda)\|_{\mathcal{K}^1_{1,r}(G_\lambda)} \leq M_2\lambda^{1/2}\|v_\lambda - \Pi(v_\lambda)\|_{H^1_r(G_\lambda)} \\
&\leq CM_2\lambda^{1/2}(h/\lambda)\|v_\lambda\|_{H^2_r(G_\lambda)} \leq CM_1M_2\lambda^{1/2}(h/\lambda)\|v_\lambda\|_{\mathcal{K}^2_{1,r}(G_\lambda)} \\
&= CM_1M_2(h\tilde{l}/\xi)\|v\|_{\mathcal{K}^2_{1,r}(G)} \leq C(\kappa)\xi^a(h/\xi)\|v\|_{\mathcal{K}^2_{a+1,r}(G)},
\end{aligned}$$

where we used the fact that the spaces $H_r^m$ and $\mathcal{K}^m_{1,r}$ are equivalent on $G_\lambda$ (Lemma 2.13), the dilation from Lemma 4.9, and the last inequality is from Lemma 2.12.

For a vertex $Q_i \notin \{r = 0\}$, the proof is similar. Note on $\mathcal{V}_i$, $\Pi v$ is actually the nodal interpolate of $v$. With the corresponding estimate in Lemmas 4.5 and 4.9, we conclude the proof by

$$\begin{aligned}
\|v - \Pi v\|_{\mathcal{K}^1_{1,r}(G)} &\leq C_1^{-1}\|v_\lambda - (\Pi v)_\lambda\|_{\mathcal{K}^1_{1,r}(G_\lambda)} \\
&= C_1^{-1}\|v_\lambda - \Pi(v_\lambda)\|_{\mathcal{K}^1_{1,r}(G_\lambda)} \leq M_2C_1^{-1}\|v_\lambda - \Pi(v_\lambda)\|_{H^1_r(G_\lambda)} \\
&\leq CM_2C_1^{-1}(h/\lambda)\|v_\lambda\|_{H^2_r(G_\lambda)} \leq CM_1M_2C_1^{-1}(h/\lambda)\|v_\lambda\|_{\mathcal{K}^2_{1,r}(G_\lambda)} \\
&\leq CM_1M_2C_1^{-1}C_2(h\tilde{l}/\xi)\|v\|_{\mathcal{K}^2_{1,r}(G)} \leq C(\kappa)\xi^a(h/\xi)\|v\|_{\mathcal{K}^2_{a+1,r}(G)}. \quad \square
\end{aligned}$$

### 4.3. Construction of the finite element spaces

In this subsection, we construct a sequence of meshes $\mathcal{T}_n$ and the finite element spaces $S_n := S(\mathcal{T}_n, 1) \subset H_r^1(\Omega) \cap \{v|_{\Gamma_0} = 0\}$ associated to the linear Lagrange triangle, such that the finite element approximations for Eq. (4) $u_n := u_{S_n} \in S_n$ satisfy

$$\|u - u_n\|_{H_1^r(\Omega)} \leq C\,\dim(S_n)^{-1/2}\|f\|_{L_r^2(\Omega)},$$

even if the solution $u \notin H_+^2$. We shall achieve this *quasi-optimal* rate of convergence by considering a suitable grading technique close to the points in $\mathcal{S}$. The proof is based on the error estimates in weighted spaces $H_r^m$ and $\mathcal{K}^m_{a,r}$, established in the previous subsections.

To be more precise, we construct the meshes $\mathcal{T}_n$ by successive refinements from an initial triangulation. Therefore, they are nested and will have the same number of triangles as the meshes obtained by the usual midpoint refinements.

From now on, we let $\eta = \min(1, |\Sigma_i|, \sqrt{\lambda_{i,1} + 1/4})$, which satisfies Theorem 3.5. We assume that in Eq. (4), the right hand side $f \in L_r^2$. Therefore, by Theorems 3.3 and 3.5, the unique solution $u$ of Eq. (4) satisfies

$$u \in \mathcal{K}^2_{a+1,+} \cap \{v|_{\Gamma_0} = 0\}, \quad \text{for } 0 \leq a < \eta.$$

We now introduce our refinement procedure.

**Definition 4.12.** Let $\kappa \in (0, 1/2]$ and $\mathcal{T}$ be a triangulation of $\Omega$ such that no two vertices of $\Omega$ belong to the same triangle of $\mathcal{T}$. Then the $\kappa$*-refinement* of $\mathcal{T}$, denoted by $\kappa(\mathcal{T})$, is obtained by dividing each edge $AB$ of $\mathcal{T}$ in two parts as follows. If neither $A$ nor $B$ is in the vertex set $\mathcal{S}$, then we divide $AB$ into two equal parts. Otherwise, if $A$ is in $\mathcal{S}$, we divide $AB$ into $AC$ and $CB$ such that $|AC| = \kappa|AB|$. This will divide each triangle of $\mathcal{T}$ into four triangles (Fig. 3).

We now introduce our sequence of meshes. Recall that $\tilde{l} > 0$ was introduced in Eq. (7), and $4\tilde{l}$ is not greater than the distance from a vertex $Q \in \mathcal{S}$ to an edge of $\Omega$ that does not contain it.

**Definition 4.13.** Suppose the initial mesh $\mathcal{T}_0$ is such that each edge in the mesh has length $\leq \tilde{l}/2$ and each point in $\mathcal{S}$ is the vertex of a triangle in $\mathcal{T}_0$. In addition, we chose $\mathcal{T}_0$ such that there is no triangle in $\mathcal{T}_0$ that contains more than one point in $\mathcal{S}$. Then we define by induction $\mathcal{T}_{n+1} = \kappa(\mathcal{T}_n)$ (see Definition 4.12).

**Remark 4.14.** Note that near the vertices, our refinement coincides with the one introduced in [16,41,31]. In addition, we may use different $\kappa$'s at different vertices as in [28] to improve the shape regularity of the triangles (see [49] for example).

We now investigate the approximation properties afforded by the triangulation $\mathcal{T}_n$ close to a point $Q_i \in \mathcal{S}$. We also fix a triangle $T \in \mathcal{T}_0$ that has $Q_i$ as a vertex. We denote by $T_{\kappa^j} = \kappa^j T \subset T$ the small triangle belonging to $\mathcal{T}_j$ that is similar to $T$ with ratio $\kappa^j$, with $Q_i$ as a vertex. Then $T_{\kappa^j} \subset T_{\kappa^{j-1}}$. Moreover, since $\kappa \leq 1/2$ and the diameter of $T$ is $\leq \tilde{l}/2$, we have $T_{\kappa^j} \subset \mathcal{V}_i, j \geq 1$, by the definition of $\mathcal{V}_i$.

Let $N$ be the level of refinements. In all the statements below, let $h \simeq 2^{-N}$, in the sense that they have comparable magnitudes. We first have the estimate on the last triangle that contains the vertex $Q_i$.

**Lemma 4.15.** *Let $0 < \kappa \leq 2^{-1/a}$, for any $0 < a < \eta$. We consider the small triangle $T_{\kappa^N} = \kappa^N T \subset T$ with vertex $Q_i$, obtained after $N$ refinements. Recall the definition of $\Pi u$ on the triangulation $\mathcal{T}_N$ for $u \in \mathcal{K}^2_{a+1,r}(\mathcal{V}_i) \cap \{v|_{\Gamma_0} = 0\}$ in (21). Then, if the vertex $Q_i \notin \{r = 0\}$, we have*

$$\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^N})};$$

*if the vertex $Q_i \in \{r = 0\}$, we have*

$$\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(U_{\kappa^N})};$$

*for $u \in \mathcal{K}^2_{a+1,r}(\mathcal{V}_i) \cap \{v|_{\Gamma_0} = 0\}$, where $h \simeq 1/2^N$ and $U_{k^N}$ is the patch associated to $T_{\kappa^N}$ in Remark 4.10, and $C$ depends on the shape regularity of $\mathcal{T}_0$ and $\kappa$.*

**Proof.** Define $u_\lambda(r, z) = u(\lambda r, \lambda z)$ with $Q_i$ as the origin. If the vertex $Q_i \in \{r = 0\}$, let $\lambda = \kappa^N$ and $U_\lambda := \lambda^{-1} U_{\kappa^N}$. Then, $T \subset U_\lambda$. Let $\chi : U_\lambda \to [0, 1]$ be a smooth function that is equal to 0 in a neighborhood of $Q_i$ and is equal to 1 at all the nodal points of $U_\lambda$ different from the vertex $Q_i$. We introduce the auxiliary function $v = \chi u_\lambda$ on $U_\lambda$. Note that $v \in H^2_r(U_\lambda)$, since $v = 0$ in the neighborhood of $Q_i$. Consequently, for $m = 0, 1, 2$,

$$\|v\|^2_{\mathcal{K}^m_{1,r}(U_\lambda)} = \|\chi u_\lambda\|^2_{\mathcal{K}^m_{1,r}(U_\lambda)} \leq C\|u_\lambda\|^2_{\mathcal{K}^m_{1,r}(U_\lambda)},$$

where $C$ depends on $m$ and the choice of the nodal points. Moreover, by the definitions of $v$ and the operator $\Pi$, we note $\Pi v = \Pi(u_\lambda) = (\Pi u)_\lambda$ on $U_\lambda$. Then,

$$\begin{aligned}
\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} &= \lambda^{1/2}\|u_\lambda - v + v - \Pi(u_\lambda)\|_{\mathcal{K}^1_{1,r}(T)} \\
&\leq \lambda^{1/2}(\|u_\lambda - v\|_{\mathcal{K}^1_{1,r}(T)} + \|v - \Pi(u_\lambda)\|_{\mathcal{K}^1_{1,r}(T)}) \\
&= \lambda^{1/2}(\|u_\lambda - v\|_{\mathcal{K}^1_{1,r}(T)} + \|v - \Pi v\|_{\mathcal{K}^1_{1,r}(T)}) \leq C\lambda^{1/2}(\|u_\lambda\|_{\mathcal{K}^1_1(T)} + \|v\|_{\mathcal{K}^2_{1,r}(U_\lambda)}) \\
&\leq C\lambda^{1/2}(\|u_\lambda\|_{\mathcal{K}^1_{1,r}(T)} + \|u_\lambda\|_{\mathcal{K}^2_{1,r}(U_\lambda)}) = C(\|u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} + \|u\|_{\mathcal{K}^2_{1,r}(U_{\kappa^N})}) \\
&\leq C\kappa^{Na}\|u\|_{\mathcal{K}^2_{a+1,r}(U_{\kappa^N})} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(U_{\kappa^N})}.
\end{aligned}$$

The first and the sixth relations above are due to Lemma 4.9; the fourth is due to Lemma 4.6 and the fact that the $\mathcal{K}^m_{1,r}$-norm and the $H^m_r$-norm are equivalent for $v$, since $v = 0$ in the neighborhood of $Q_i$; the seventh is based on Lemma 2.12 and the fact that the size of $U_{\kappa^N}$ is comparable with $\kappa^N$.

The estimate for a vertex $Q_i \notin \{r = 0\}$ is similar, but requires another inequality in Lemma 4.9 and the estimate in Lemma 4.5, since $\Pi u$ is actually the nodal interpolation. Let $\chi : T \to [0, 1]$ be a smooth function that is equal to 0 in a neighborhood of $Q_i$ and is equal to 1 at all the other nodal points of $T$. Let $\lambda = \kappa^N$ and $v = \chi u_\lambda$. Then,

$$\begin{aligned}
\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} &\leq C_1^{-1}\|u_\lambda - v + v - (\Pi u)_\lambda\|_{\mathcal{K}^1_{1,r}(T)} \\
&\leq C_1^{-1}(\|u_\lambda - v\|_{\mathcal{K}^1_{1,r}(T)} + \|v - \Pi v\|_{\mathcal{K}^1_{1,r}(T)}) \leq C_1^{-1}C(\|u_\lambda\|_{\mathcal{K}^1_{1,r}(T)} + \|v\|_{\mathcal{K}^2_{1,r}(T)}) \\
&\leq C_1^{-1}C(\|u_\lambda\|_{\mathcal{K}^1_{1,r}(T)} + \|u_\lambda\|_{\mathcal{K}^2_{1,r}(T)}) \leq C_1^{-1}CC_2(\|u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^N})} + \|u\|_{\mathcal{K}^2_{1,r}(T_{\kappa^N})}) \\
&\leq C\kappa^{Na}\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^N})} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^N})},
\end{aligned}$$

where $C_1$ and $C_2$ are from Lemma 4.9. $\square$

We now combine the estimates on $T_{\kappa^N}$ from Lemma 4.15 with the estimates on the sets of the form $T_{\kappa^j} \setminus T_{\kappa^{j+1}}$ from Lemma 4.11 to obtain the following estimate on a triangle $T \in \mathcal{T}_0$ that has a vertex in $\mathcal{S}$.

**Proposition 4.16.** *Let $h \simeq 1/2^N$ and $0 < \kappa \leq 2^{-1/a}$ for $0 < a < \eta$. Then, after $N$ $\kappa$-refinements on $T$, there exists a constant $C > 0$, such that*

$$\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T)} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(T\cup U_{\kappa^N})},$$

*for all $u \in \mathcal{K}^2_{a+1,r} \cap \{v|_{\Gamma_0} = 0\}$.*

**Proof.** Definition 4.13 shows that the mesh on $T_{\kappa^{j-1}} \setminus T_{\kappa^j}$ satisfies the assumption in Lemma 4.11 and has the size $\simeq \kappa^{j-1}2^{j-1-N}$. Using the notation of Lemma 4.11, we have $\xi = O(\kappa^{j-1})$ on $T_{\kappa^{j-1}} \setminus T_{\kappa^j}$. Therefore,

$$\begin{aligned}
\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(T_{\kappa^{j-1}}\setminus T_{\kappa^j})} &\leq C\kappa^{(j-1)a}(\kappa^{j-1}2^{j-1-N}/\kappa^{j-1})\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^{j-1}}\setminus T_{\kappa^j})} \\
&\leq C2^{-(j-1)}2^{-N+(j-1)}\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^{j-1}}\setminus T_{\kappa^j})} = C2^{-N}\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^{j-1}}\setminus T_{\kappa^j})} \\
&\leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(T_{\kappa^{j-1}}\setminus T_{\kappa^j})},
\end{aligned}$$

where $C$ depends on $\kappa$ and $\mathcal{T}_0$, but not on the subset $T_{\kappa^{j-1}} \setminus T_{\kappa^j}$. We then complete the proof by adding up the error estimates on all the subsets $T_{\kappa^{j-1}} \setminus T_{\kappa^j}$, $1 \leq j \leq N$, and on $T_{\kappa^N}$ from Lemma 4.15. □

**Remark 4.17.** Denote by $\mathbb{T}$, the union of all the initial triangles that contain vertices of $\Omega$. Then $\mathbb{T}$ is a neighborhood of $\mathcal{S}$. Note that the union of the patches $U_{\kappa^N}$ for the vertices on the $z$-axis is a subset of $\mathbb{T}$. Therefore, summing up the estimates in Proposition 4.16 over all the triangles in $\mathbb{T}$ gives $\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(\mathbb{T})} \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}(\mathbb{T})}$, as long as $\kappa$ is chosen appropriately.

We state the main result on the convergence of numerical solutions on our meshes.

**Theorem 4.18.** *Let $0 < a < \eta$ and $0 < \kappa \leq 2^{-1/a}$. Let $\mathcal{T}_n$ be obtained from the initial triangulation by $n$ $\kappa$-refinements (Definition 4.13). Let $u$ be the solution of Eq. (4). Denote by $S_n := S_n(\mathcal{T}_n, 1) \subset H^1_r \cap \{v|_{\Gamma_0} = 0\}$ the finite element space associated to the linear Lagrange triangle and by $u_n \in S_n$ the finite element solution defined by Eq. (16). Then, there exists $C > 0$ depending on the domain and the initial triangulation, such that*

$$\|u - u_n\|_{\mathcal{K}^1_{1,r}} \leq Ch\|f\|_{L^2_r}, \quad \text{for } h \simeq 2^{-n}, \ \forall f \in L^2_r.$$

**Proof.** Let $\mathbb{T}$ be the union of initial triangles that contain vertices of $\Omega$ as in Remark 4.17. Recall from Theorem 3.3 that $\|u\|_{\mathcal{K}^2_{a+1,r}} \leq \|u\|_{\mathcal{K}^2_{a+1,+}} \leq C\|f\|_{L^2_r}$. We obtain

$$\begin{aligned}
\|u - u_n\|_{\mathcal{K}^1_{1,r}} &\leq C\|u - \Pi u\|_{\mathcal{K}^1_{1,r}} \leq C\big(\|u - \Pi u\|_{\mathcal{K}^1_{1,r}(\Omega\setminus\mathbb{T})} + \|u - \Pi u\|_{\mathcal{K}^1_{1,r}(\mathbb{T})}\big) \\
&\leq Ch\big(\|u\|_{\mathcal{K}^2_{1,r}(\Omega)} + \|u\|_{\mathcal{K}^2_{a+1,r}(\mathbb{T})}\big) \leq Ch\|u\|_{\mathcal{K}^2_{a+1,r}} \leq Ch\|f\|_{L^2_r}.
\end{aligned}$$

The first inequality is based on Lemma 4.1 and the third inequality is based on Lemmas 4.5 and 4.6, and Proposition 4.16. □

Then, as a direct result of the theorem above, we have the following *quasi-optimal* convergence rate for the finite element solution.

**Corollary 4.19.** *Let $0 < a < \eta$ and $0 < \kappa \leq 2^{-1/a}$. Using the notation and assumptions in Theorem 4.18, we have that $u_n \in S(\mathcal{T}_n, 1)$ satisfies*

$$\|u - u_n\|_{H^1_r} \leq C\dim(S_n)^{-1/2}\|f\|_{L^2_r},$$

*for a constant $C$ independent of $f$ and $n$.*

**Proof.** Let $\mathcal{T}_n$ be the triangulation of $\Omega$ after $n\kappa$-refinements. Then, the number of triangles is $O(4^n)$ based on the construction of triangles in different levels. Therefore, the dimension of $S_n$, $\dim(S_n) \simeq 4^n$, for Lagrange triangles. Thus, from Theorem 4.18, the following estimates are obtained,

$$\|u - u_n\|_{H^1_r} \leq \|u - u_n\|_{\mathcal{K}^1_{1,r}} \leq Ch\|f\|_{L^2_r} \leq C2^{-n}\|f\|_{L^2_r} \leq C\dim(S_n)^{-1/2}\|f\|_{L^2_r}.$$

Then, the proof is complete. □

**Remark 4.20.** Note that the "optimal" range for $\kappa$ is $(0, 2^{-1/\eta})$, in which the finite element solution will have the *quasi-optimal* rate of convergence. We also notice that a small $\kappa$ results in thin triangles that may lead to a large constant $C$ (see [50]) in the estimate. Therefore, a good choice of $\kappa$ is a value close to the upper bound of the "optimal" range, such that we have both the *quasi-optimal* rate of convergence for the finite element solution and a better shape regularity of the triangulation.

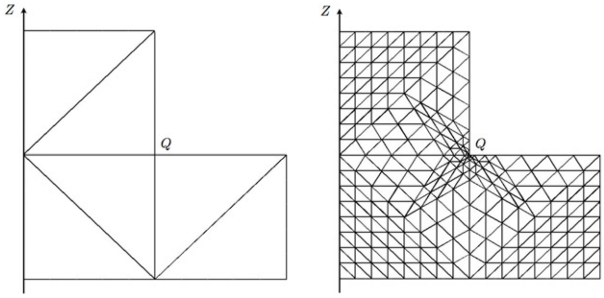


**Fig. 4.** An initial triangulation of the L-shape domain $\Omega_1$ (left); the mesh after 3 successive $\kappa$-*refinements* (right), $\kappa = 0.2$.

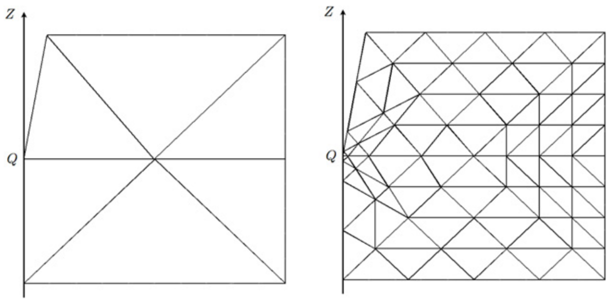


**Fig. 5.** An initial triangulation of the domain $\Omega_2$, $\angle Q = 170°$ (left); the mesh after 2 successive $\kappa$-*refinements* (right), $\kappa = 0.2$.

## 5. Numerical illustrations

We present some numerical results that illustrate the effectiveness of our meshing techniques for solving the axisymmetric boundary value problem (4). These tests convincingly show that our sequence of meshes achieves the *quasi-optimal* rates of convergence in the energy norm $\| \cdot \|_{H^1_r}$.

We shall see that a choice of $\kappa$ in the acceptable range $(0, 2^{-1/\eta})$ yields *quasi-optimal* rates of convergence, whereas a choice of $\kappa$ out of this range will not give the same convergence rates. Since $\kappa$ decreases as $\eta$ decreases, a good determination of $\eta$ will certainly help us to choose $\kappa$, such that we obtain the *quasi-optimal* convergence rate, while avoid *thin* triangles if possible.

### *5.1. Numerical tests*

We here consider Eq. (4), with the right hand side $f = 1$, on two domains, $\Omega_1$ (Fig. 4) and $\Omega_2$ (Fig. 5), to illustrate our treatments for vertices away from the $z$-axis and for vertices on the $z$-axis. See Fig. 6 for numerical solutions on these domains.

$\Omega_1$ is an L-shape domain with an edge on the $z$-axis (Fig. 4). Our theoretical results show that on $\Omega_1$, the solution of (4) is not in $H^2_+$ at the re-entrant corner with vertex $Q$. Therefore, a special $\kappa$-*refinement* is needed near this vertex to ensure the convergence rate predicted in Corollary 4.19.

To be more precise, from the theory we developed in Section 3 (Eq. (15)), we can take any value for $a$, such that $0 < a < \eta = \pi/1.5\pi \approx 0.667$, which gives $2^{-1/a} < 2^{-1/\eta} \approx 0.354$. Hence, for any $\kappa < 0.354$, we expect that the $\kappa$-*refinement* near $Q$ will lead to the *quasi-optimal* convergence rate for the finite element solution. In particular, since the space $H^m_r$ is equivalent to the usual Sobolev space $H^m$ near $Q$ on $\Omega_1$, a more accurate a prior estimate [21] gives $u \in H^s_r(\Omega_1)$ for $s < 1 + \eta \approx 1.667$, where $H^s_r$ is defined by interpolation [1].

$\Omega_2$ is a polygon with one side on the $z$-axis (Fig. 5), where the interior angle of the corner with $Q$ as the vertex is 170°. For vertices on the $z$-axis, the values of $\eta$ for appropriate meshes follow another formula $\eta = \sqrt{\lambda_1 + 1/4}$, where $\lambda_1$ is the smallest real eigenvalue of the Laplace–Beltrami operator discussed in Section 3. This gives $\eta \approx 0.7 < 1$, for $\angle Q = 170°$, which means the solution is not in $H^2_+$ near this vertex. Therefore, we may choose any $\kappa < 2^{-1/\eta} \approx 0.372$ near the vertex $Q$, in order to get the *quasi-optimal* rate of convergence. It is also interesting to note that given the same interior angle, the singularities near the vertices on the $z$-axis are stronger than those near the vertices away from the $z$-axis.

Based on the calculation of the parameter $a$ for each vertex, on both $\Omega_1$ and $\Omega_2$, the solutions are in $H^2_+$, except in the neighborhoods of the vertex $Q$. Therefore, we use the usual midpoint refinements near vertices different from $Q$ on both domains.

**Table 1**
Comparison of convergence rates for different values of $\kappa$.

| $j \setminus \kappa$ | Convergence history on $\Omega_1$ | | | | | Convergence history on $\Omega_2$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $e : 0.1$ | $e : 0.2$ | $e : 0.3$ | $e : 0.4$ | $e : 0.5$ | $e : 0.1$ | $e : 0.2$ | $e : 0.3$ | $e : 0.4$ | $e : 0.5$ |
| 3 | 0.929 | 0.942 | 0.935 | 0.898 | 0.836 | 0.866 | 0.878 | 0.878 | 0.855 | 0.797 |
| 4 | 0.970 | 0.976 | 0.968 | 0.923 | 0.825 | 0.902 | 0.920 | 0.916 | 0.881 | 0.796 |
| 5 | 0.988 | 0.990 | 0.981 | 0.928 | 0.795 | 0.936 | 0.953 | 0.942 | 0.893 | 0.781 |
| 6 | 0.996 | 0.996 | 0.986 | 0.926 | 0.763 | 0.967 | 0.976 | 0.960 | 0.901 | 0.764 |
| 7 | 0.999 | 0.998 | 0.990 | 0.923 | 0.735 | 0.986 | 0.989 | 0.972 | 0.906 | 0.749 |
| 8 | 1.000 | 0.999 | 0.992 | 0.919 | 0.713 | 0.995 | 0.995 | 0.980 | 0.909 | 0.736 |
| 9 | 1.000 | 1.000 | 0.994 | 0.915 | 0.698 | 0.999 | 0.998 | 0.986 | 0.911 | 0.726 |

$e = \log_2\left(\frac{|u_j - u_{j-1}|_{H^1_r}}{|u_{j+1} - u_j|_{H^1_r}}\right)$.

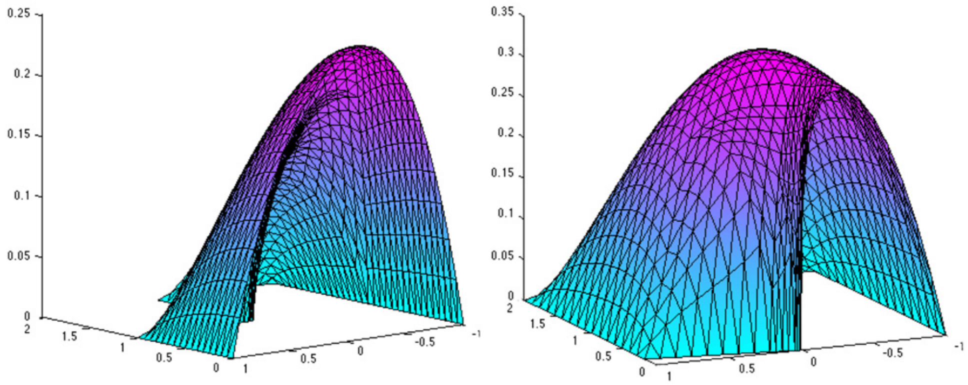

**Fig. 6.** The numerical solution on $\Omega_1$ (left); the numerical solution on $\Omega_2$ (right).

Table 1 lists the convergence rates of the finite element solutions for the axisymmetric problem on $\Omega_1$ and $\Omega_2$, respectively, for triangulations with different values of $\kappa$ near the vertex $Q$. These results verify our theoretical prediction: the *quasi-optimal* convergence rates can be obtained for $\kappa < 0.354$ on the L-shape domain $\Omega_1$ and obtained for $\kappa < 0.372$ on the polygon $\Omega_2$.

The left most column (values of $j$) in Table 1 represents the refinement levels. Let $u_j$ be the finite element solution on the mesh after $j$ refinements. The quantities printed out in other columns in the table are the convergence rates defined by

$$e = \log_2\left(\frac{|u_j - u_{j-1}|_{H^1_r}}{|u_{j+1} - u_j|_{H^1_r}}\right),$$

which is a reasonable approximation of the exact convergence rate.

Recall $h \simeq (1/2)^j$ for the mesh after $j$ refinements. Then, we see that on $\Omega_1$, for appropriate graded meshes ($\kappa < 0.354$), the convergence rates are $h^1$, while on uniform meshes ($\kappa = 0.5$), the convergence rates have slowed down to $h^{0.7}$, which is very close to the theoretical rate 0.667 from our estimates above and seems to get closer and closer to 0.667.

On $\Omega_2$, from the discussion above, we have found that the convergence rates of the discrete solutions should be *quasi-optimal* ($h^1$) as long as $\kappa < 0.372$, which matches the numerical results in Table 1 perfectly. In addition, the convergence rates in the column for $\kappa = 0.3$ and $\kappa = 0.4$ have a large gap, indicating the critical value of $\kappa$ for the good convergence rates lies between 0.3 and 0.4, which, once again, verifies the theory.

### 5.2. Summary

As a brief summary, we have tested our method for the model problem on two domains, $\Omega_1$ and $\Omega_2$, for singularities of different types. All the results in Table 1 convincingly show that the theoretical rate of convergence is consistent with our numerical computations. Therefore, for the axisymmetric problem (4), with the regularity of the solution determined in terms of weighted Sobolev spaces $\mathcal{K}^m_{a,r}$, the numerical solutions have the convergence rate $\dim(S_n)^{-1/2}$, on correctly graded meshes. Standard quasi-uniform meshes exhibit rates of convergence that are less than optimal when the solution fails to be in $H^2_+$ (which happens if $\eta < 1$).

The finest mesh in our numerical tests above is obtained after 10 successive refinements of the coarsest mesh and has roughly $2^{23} \approx 8 \times 10^6$ elements. The preconditioned conjugate gradient (PCG) method is used to solve the resulting system of algebraic equations. Besides the application on the FEM, our regularity results may be useful for the analysis of the generalized finite element method (GFEM) and the multigrid method (MG) for axisymmetric problems. See [51–56,6,7,57,58] for relevant discussions on these subjects.

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