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Double Layer Potentials on Polygons and Pseudodifferential Operators on Lie Groupoids

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Abstract. We use an approach based on pseudodifferential operators on Lie groupoids to study the double layer potentials on plane polygons. Let Ω be a simply connected polygon in $\mathbb{R}^2$. Denote by K the double layer potential operator on Ω associated with the Laplace operator Δ. We show that the operator K belongs to the groupoid C^*-algebra that the first named author has constructed in an earlier paper (Carvalho and Qiao in Cent Eur J Math 11(1):27–54, 2013). By combining this result with general results in groupoid C^*-algebras, we prove that the operators $\pm I + K$ are Fredholm between appropriate weighted Sobolev spaces, where I is the identity operator. Furthermore, we establish that the operators $\pm I + K$ are invertible between suitable weighted Sobolev spaces through techniques from Mellin transform. The invertibility of these operators implies a solvability result in weighted Sobolev spaces for the interior and exterior Dirichlet problems on Ω.

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1. Introduction

Potential theory can be traced back to the works of Lagrange, Laplace, Poisson, Gauss, and others [42], and plays a fundamental role in many real-world problems, especially in physics. Many works are devoted to the method of layer potentials. We mention here a few monographs, beginning with the books by Courant and Hilbert [14], Folland [21], Hsiao and Wendland [23],

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Kress [32], Mclean [42], and Taylor [69]. These monographs give a rather complete account of the theory of layer potential operators on smooth domains. Let us also mention the paper [20], which includes some results on $\mathcal{C}^1$-domains.

There are also many papers devoted to the method of layer potentials on non-smooth domains, which can be roughly divided into two categories: one devoted to Lipschitz domains and the other to polyhedral domains.

The case of Lipschitz domains, by far the most studied among the class of non-smooth domains, is also fairly well understood. We mention the papers of Jerrison and Kenig [24,25], Kenig [28], and Verchota [70] for relevant results on domains in the Euclidean space. In the works of Mitrea and Mitrea [47], Mitrea and Mitrea [49], Mitrea and Taylor [51,52], and Kohr et al. [29], the method of layer potentials is applied to Lipschitz domains on manifolds. See also Costabel's paper [12] for an introduction to the method of layer potentials, in which more elementary methods are applied.

We are interested in nonsmooth domains, especially in polyhedral domains. By comparison, much fewer works were dedicated to this case. We mention however the papers of Ammann et al. [2], Lewis and Parenti [35], and Mitrea [48] for results on polygonal domains. The works of Elschner [18], Fabes et al. [19], Angell et al. [6], Medkova [43], and Verchota and Vogel [71] deal with the case of polyhedral domains in three and four dimensions. The paper [41] concentrates on polyhedral domains and domains with cracks. See [27] for the related case of interface problems.

In addition, boundary value problems on domains with conical points were studied by many authors. We mention in this regard the work of Kondratiev [30], the papers of Kapanadze and Schulze [26], Lewis and Parenti [35], Li et al. [36], Mazzeo and Melrose [40] and Melrose [44], and Schröhe and Schulze [64,65]. See also the books of Egorov and Schulze [17], Kozlov et al. [31], Mazya and Rossmann [39], Melrose [45,46], Schulze [66], Schulze et al. [67], and Sauter and Schwab [63]. Many of these works are devoted to constructing suitable algebras of pseudodifferential operators on conical manifolds. See also the paper [1,4,5,15,16] using groupoids to construct algebras of pseudodifferential operators on singular spaces, and [58,68] for some related constructions.

In this paper, we study the double layer potential operator K associated with the Laplace operator on a plane polygon. Let $\Omega \subset \mathbb{R}^2$ be a (regular) open bounded domain. Consider the interior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \phi & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

and the exterior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega^c \\ u|_{\partial\Omega} = \phi & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where Ω^c denotes the complement of $\overline{\Omega}$, i.e., $\Omega^c = \mathbb{R}^2 \backslash \overline{\Omega}$.

For $\psi \in \mathcal{C}_c^\infty(\partial\Omega)$, define the double layer potential

$$u(x) = -\, c_2 \int_{\partial\Omega} \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\psi(y)d\sigma(y), \qquad (x \in \mathbb{R}^2 \backslash \partial\Omega),$$

where $\nu(y)$ is the exterior unit normal to a point $y \in \partial\Omega$ and c_2 is a constant. Conventionally c_2 is taken to be $\frac{1}{\pi}$ in this paper.

let $u_-(x)$ and $u_+(x)$ denote the limits of $u(z)$ as $z \to x$ nontangentially from $z \in \Omega$ and $z \in \mathbb{R}^2\backslash\overline{\Omega}$, respectively. The classical results [13,21,69] on double layer potentials give that for (a.e.) $x \in \partial\Omega$, we have

1. $u_-(x) = \psi(x) + K\psi(x), i.e., u_- = (I+K)\psi$;
2. $u_+(x) = -\,\psi(x) + K\psi(x), i.e., u_+ = (-I+K)\psi$, where

$$K\psi(x) = \int_{\partial\Omega} k(x,y)\psi(y)d\sigma(y),$$

with $k(x,y) = -c_2\dfrac{(x-y)\cdot\nu(y)}{|x-y|^2}$.

Hence, the interior and exterior Dirichlet problems are reduced to solving boundary integral equations $(I+K)\psi = \phi$ and $(-I+K)\psi = \phi$, respectively, where ϕ is the given function on the boundary $\partial\Omega$ and ψ is the unknown function on $\partial\Omega$.

In general, the double layer potential method works for (regular) domains in $\mathbb{R}^n$, $n \geqslant 2$. For instance, in [21,69], it is shown that if the domain $\Omega \subset \mathbb{R}^n$ has $\mathcal{C}^2$ boundary $\partial\Omega$, then the double layer potential operator K is compact on $L^2(\partial\Omega)$ (and $H^m(\partial\Omega)$). Hence operators $\pm I + K$ are Fredholm of index zero. Therefore, the solvability of the interior and exterior Dirichlet problems is equivalent to injectivity or surjectivity of $\pm I + K$. If the boundary $\partial\Omega$ is not $\mathcal{C}^2$, the operator K is no longer compact (see [18,19,21,22,30,32,35,47,48,50,70]). However, we can still hope that $\pm I + K$ are Fredholm operators on appropriate function spaces on the boundary. Recently, Perfekt and Putinar have studied the essential spectrum of the double layer potential operator K on a planar domain with corners and give a complete result of the essential spectrum of K on the Sobolev space of order $\frac{1}{2}$ along the boundary [59,60].

From the pseudodifferential operator point of view, if the boundary $\partial\Omega$ is smooth, the double layer potential operator K is a pseudodifferential operator of order -1 on the boundary [69]. The survey [38] emphasizes the importance of understanding the algebra of pseudodifferential operators on singular spaces. In this paper, we use a groupoid approach to construct algebras of pseudodifferential operators (and C^*-algebras) on polygons in the spirit of [3,56], specifically in the framework of Fredholm groupoids [9,10]. Then we show that the double layer potential operator K lies in this groupoid C^*-algebra. From this result, we demonstrate that the operators $\pm I + K$ are Fredholm between appropriate weighted Sobolev spaces on the boundary of the domain. Consequently, we use techniques from Mellin transform to prove that the operators $\pm I + K$ are isomorphic between suitable weighted Sobolev spaces. This implies a solvability result in weighted Sobolev spaces for the interior and exterior Dirichlet problems on Ω. It is also possible to extend our method to solve interior and exterior Neumann problems.

Our main results are as follows. Let Ω be a simply connected polygon in R^2 with vertices $P_1, P_2, \ldots, P_n$. Denote by θ_i the interior angle at vertex P_i. Throughout the paper, we always assume that Ω be a simply connected polygon in R^2.

Let $\mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$ be the Sobolev space on $\partial\Omega$ with weight r_Ω and index a (see Sect. 2). Define

$$\theta_0 := \min\left\{\frac{\pi}{\theta_1}, \frac{\pi}{2\pi-\theta_1}, \frac{\pi}{\theta_2}, \frac{\pi}{2\pi-\theta_2}, \ldots, \frac{\pi}{\theta_n}, \frac{\pi}{2\pi-\theta_n}\right\}.$$

Clearly, $\frac{1}{2} < \theta_0 < 1$. Then we have

Theorem 1.1. *For $a \in (-\theta_0, 1/2)$ and $m \geq 0$, the operators*

$$\pm I + K : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$$

are isomorphisms.

The paper is organized as follows. In Sect. 2, we review weighted Sobolev spaces on plane polygons and briefly recall desingularization of polygons. In Sect. 3, we collect basic concepts of pseudodifferential operators on Lie groupoids. Then, we give an explicit analysis on the double layer potential operator K associated to a plane sector and discuss its connection to a (smooth invariant) family of operators on certain Lie groupoid in Sect. 4. Section 5 contains the proofs our main result. Namely, the operators $\pm I + K$ are isomorphisms on weighted Sobolev spaces with suitable weights. We end with concluding remarks in Sect. 6.

2. Weighted Sobolev Spaces on Polygons and Desingularization

Let Ω be a plane polygon and $m \in \mathbb{Z}_{\geqslant 0}$. Let α be a multi-index, and r_Ω be the weight function which is equivalent to the distance function to the vertices of Ω (see [7] for details). We define the mth Sobolev space on Ω with weight r_Ω and index a by

$$\mathcal{K}^m_a(\Omega) = \{u \in L^2_{\text{loc}}(\Omega) \mid r_\Omega^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq m\}.$$

The norm on $\mathcal{K}^m_a(\Omega)$ is

$$||u||^2_{\mathcal{K}^m_a(\Omega)} := \sum_{|\alpha|\leq m} ||r_\Omega^{|\alpha|-a}\partial^\alpha u||^2_{L^2(\Omega,dx)}.$$

By Theorem 5.6 in [7], this norm is equivalent to

$$||u||^2_{m,a} := \sum_{|\alpha|\leq m} ||r_\Omega^{-a}(r_\Omega\partial)^\alpha u||^2_{L^2(\Omega,dx)},$$

where $(r_\Omega\partial)^\alpha = (r_\Omega\partial_1)^{\alpha_1}(r_\Omega\partial_2)^{\alpha_2}\cdots(r_\Omega\partial_n)^{\alpha_n}$.

Clearly, we have that

$$r^t_\Omega\mathcal{K}^m_a(\Omega) \cong \mathcal{K}^m_{a+t}(\Omega).$$

In general, this isomorphism may not be an isometry.

In [7], there is a standard procedure to desingularize Ω. Denote by $\Sigma(\Omega)$ the desingularization of Ω, which is a Lie manifold with boundary. The space $L^2(\Sigma(\Omega))$ is defined by using the volume element of a compatible metric with the Lie structure at infinity on $\Sigma(\Omega)$. A compatible metric is $r_\Omega^{-2} g_e$, where g_e is the Euclidean metric. Then Sobolev spaces $H^m(\Sigma(\Omega))$ are defined by using $L^2(\Sigma(\Omega))$.

Proposition 2.1. *We have, for all $m \in \mathbb{Z}$,*

$$\mathcal{K}_1^m(\Omega) \cong H^m(\Sigma(\Omega), g),$$

where the metric $g = r_\Omega^{-2} g_e$.

Proof. The result follows from Proposition 5.7 in [7]. □

The identification given above allows us to define weighted Sobolev spaces on the boundary $\mathcal{K}_a^m(\partial\Omega)$. For more details, see [7].

Proposition 2.2. *For $m \in \mathbb{Z}_{\geqslant 0}$, we have the following identification:*

$$\mathcal{K}_{\frac{1}{2}}^m(\partial\Omega) \cong H^m(\partial'\Sigma(\Omega)),$$

where $\Sigma(\Omega)$ is the desingularization of Ω and $\partial'\Sigma(\Omega)$ is the union of hyperfaces which are not at infinity.

Proof. The result follows from Definition 5.8 in [7]. □

Therefore, we have the following identifications for the weighted Sobolev spaces both on Ω and on the boundary $\partial\Omega$.

Proposition 2.3. *We have, for all $m \in \mathbb{Z}$,*

$$\mathcal{K}_1^m(\Omega) \cong H^m(\Omega, g), \quad \text{and} \quad \mathcal{K}_{\frac{1}{2}}^m(\partial\Omega) \cong H^m(\partial\Omega, g),$$

where the metric $g = r_\Omega^{-2} g_e$.

3. Pseudodifferential Operators on Lie Groupoids

3.1. Lie Groupoids

In this subsection, we review some basic facts on Lie groupoids. We begin with the definition of groupoids.

Definition 3.1. A *groupoid* is a small category $\mathcal{G}$ in which each arrow is invertible.

Let us make this definition more precise [8,34,37,53,62]. A groupoid $\mathcal{G}$ consists of two sets: one of objects (or units) $\mathcal{G}_0$ and the other of arrows $\mathcal{G}_1$. Usually we shall identify $\mathcal{G} = \mathcal{G}_1$, denote $M := \mathcal{G}_0$, and use the notation $\mathcal{G} \rightrightarrows M$. First of all, to each arrow $g \in \mathcal{G}$ we associate two units: its domain $d(g)$ and its range $r(g)$, i.e., $d, r : \mathcal{G} \to M$. Then we define the set of composable pairs

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} \mid d(g) = r(h)\}.$$

The multiplication $\mu : \mathcal{G}^{(2)} \to \mathcal{G}^{(2)}$ is given by $\mu(g, h) = gh$, and it is associative. Moreover, we have an injective map $u : M \to \mathcal{G}$, where $u(x)$ is the identity arrow of an object $x \in M$. The inverse of an arrow is denoted by $g^{-1} = \iota(g)$. We can write (in [53])

$$\mathcal{G}^{(2)} \xrightarrow{\mu} \mathcal{G} \xrightarrow{\iota} \mathcal{G} \overset{d}{\underset{r}{\rightrightarrows}} M \xrightarrow{u} \mathcal{G}.$$

A groupoid $\mathcal{G}$ is therefore completely determined by the sets M, $\mathcal{G}$ and the structural maps d, r, μ, u, ι. The structural maps satisfy the following properties:

1. $d(hg) = d(g)$, $r(hg) = r(h)$,
2. $k(hg) = (kh)g$
3. $u(r(g))g = g = gu(d(g))$, and
4. $d(g^{-1}) = r(g)$, $r(g^{-1}) = d(g)$, $g^{-1}g = u(d(g))$, and $gg^{-1} = u(r(g))$

for any $k, h, g \in \mathcal{G}_1$ with $d(k) = r(h)$ and $d(h) = r(g)$. The following definition is taken from [34].

Definition 3.2. A *Lie groupoid* is a groupoid

$$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, d, r, \mu, u, \iota)$$

such that $M := \mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds, possibly with corners, with M Hausdorff, the structural maps d, r, μ, u, and ι are smooth and the domain map d is a submersion (of manifolds with corners).

Remark 3.3. In general, the space $\mathcal{G}_1$ may not be Hausdorff. However, since d is a submersion, it follows that each fiber $\mathcal{G}_x := d^{-1}(x)$ (respectively $\mathcal{G}^x := r^{-1}(x)$) is a smooth manifold without corners, see [34,57], hence it is Hausdorff. Note that the groupoids in this paper will be Hausdorff.

3.2. Pseudodifferential Operators and Groupoid C^*-Algebras

We recall briefly the construction of the space of pseudodifferential operators associated to a Lie groupoid $\mathcal{G}$ with units M [33,34,54,55,58]. The dimension of M is $n \geq 1$.

Let $P = (P_x)$, $x \in M$ be a smooth family of pseudodifferential operators acting on $\mathcal{G}_x$. We say that P is *right invariant* if $P_{r(g)}U_g = U_g P_{d(g)}$, for all $g \in \mathcal{G}$, where

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}), \ \ (U_g f)(g') = f(g'g).$$

Let k_x be the distributional kernel of P_x, $x \in M$. Note that the support of P

$$\mathrm{supp}(P) := \overline{\bigcup_{x \in M} \mathrm{supp}(k_x)} \subset \{(g, g'),\ d(g) = d(g')\} \subset \mathcal{G} \times \mathcal{G}$$

since $\mathrm{supp}(k_x)$ is contained in $\mathcal{G}_x \times \mathcal{G}_x$. Let $\mu_1(g', g) := g'g^{-1}$. The family $P = (P_x)$ is called *uniformly supported* if its *reduced support* $\mathrm{supp}_\mu(P) := \mu_1(\mathrm{supp}(P))$ is a compact subset of $\mathcal{G}$.

Definition 3.4. The space $\Psi^m(\mathcal{G})$ of *pseudodifferential operators of order* m *on a Lie groupoid* $\mathcal{G}$ with units M consists of smooth families of pseudodifferential operators $P = (P_x)$, $x \in M$, with $P_x \in \Psi^m(\mathcal{G}_x)$, which are uniformly supported and right invariant.

We also denote $\Psi^\infty(\mathcal{G}) := \bigcup_{m\in\mathbb{R}} \Psi^m(\mathcal{G})$ and $\Psi^{-\infty}(\mathcal{G}) := \bigcap_{m\in\mathbb{R}} \Psi^m(\mathcal{G})$. We then have a representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ (or on $\mathcal{C}^\infty(M)$, on $L^2(M)$, or on Sobolev spaces), called the *vector representation* uniquely determined by the equation

$$(\pi(P)f)\circ r := P(f\circ r),$$

where $f \in \mathcal{C}_c^\infty(M)$ and $P = (P_x) \in \Psi^m(\mathcal{G})$.

Remark 3.5. If $P \in \Psi^{-\infty}(\mathcal{G})$, then P identifies with the convolution with a smooth, compactly supported function, hence $\Psi^{-\infty}(\mathcal{G})$ identifies with the convolution algebra $\mathcal{C}_c^\infty(\mathcal{G})$. In particular, we can define

$$||k_P||_{I,d} := \sup_{x\in M} \int_{\mathcal{G}_x} |k_P(g^{-1})|\, d\mu_x(g), \quad ||k_P||_{I,r} := \sup_{x\in M} \int_{\mathcal{G}^x} |k_P(g^{-1})|\, d\mu_x(g),$$

and

$$\|P\|_{L^1(\mathcal{G})} := \max\{\, ||k_P||_{I,d},\, ||k_P||_{I,r}\,\}.$$

The space $L^1(\mathcal{G})$ is defined to be the completion of $\Psi^{-\infty}(\mathcal{G}) \simeq \mathcal{C}_c^\infty(\mathcal{G})$ in the norm $||\cdot||_{L^1\mathcal{G}}$.

For each $x \in M$, there is an interesting family of representation of $\Psi^\infty(\mathcal{G})$, the *regular representations* π_x on $\mathcal{C}_c^\infty(\mathcal{G}_x)$, defined by $\pi_x(P) = P_x$. It is clear that if $P \in \Psi^{-n-1}(\mathcal{G})$

$$\|\pi_x(P)\|_{L^2(\mathcal{G}_x)} \leq \|P\|_{L^1}.$$

The *reduced C^*–norm* of P is defined by

$$\|P\|_r = \sup_{x\in M} \|\pi_x(P)\| = \sup_{x\in M} \|P_x\|,$$

and the *full norm* of P is defined by

$$\|P\| = \sup_\rho \|\rho(P)\|,$$

where ρ varies over all bounded representations of $\Psi^0(\mathcal{G})$ satisfying

$$\|\rho(P)\| \leqslant \|P\|_{L^1(\mathcal{G})} \quad \text{for all} \quad P \in \Psi^{-\infty}(\mathcal{G}).$$

Definition 3.6. Let $\mathcal{G}$ be a Lie groupoid and $\Psi^\infty(\mathcal{G})$ be as above. We define $C^*(\mathcal{G})$ (respectively, $C_r^*(\mathcal{G})$) to be the completion of $\Psi^{-\infty}(\mathcal{G})$ in the norm $\|\cdot\|$ (respectively, $\|\cdot\|_r$). If $\|\cdot\|_r = \|\cdot\|$, that is, if $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$, we call $\mathcal{G}$ *metrically amenable.*

We give some examples of Lie groupoids.

Example 3.7 (Manifolds with corners). A manifold (with corners) M may be viewed as a Lie groupoid, by taking both the object and morphism sets to be M, and the domain and range maps to be the identity map $M \to M$, and $\Psi^\infty(M) = \mathcal{C}_c^\infty(M)$.

Example 3.8 (Lie groups). Every Lie group G can be regarded as a Lie groupoid $\mathcal{G} = G$ with space of units $M = \{e\}$, the unit of G. And $\Psi^m(\mathcal{G})$ is the algebra of properly supported and invariant pseudodifferential operators on G.

Example 3.9 (Pair groupoid). Let M be a smooth manifold. Let

$$\mathcal{G} = M \times M \qquad \mathcal{G}_0 = M,$$

with structure maps $d(m_1, m_2) = m_2$, $r(m_1, m_2) = m_1$, $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$, $u(m) = (m, m)$, and $\iota(m_1, m_2) = (m_2, m_1)$. Then $\mathcal{G}$ is a Lie groupoid, called *the pair groupoid* of M. According to the definition, a pseudodifferential operator P belongs to $\Psi^m(\mathcal{G})$ if and only if the family $P = (P_x)_{x\in M}$ is constant. Hence we obtain $\Psi^m(\mathcal{G}) = \Psi^m_{\text{comp}}(M)$. Also, an important result is that $C^*(\mathcal{G}) \cong \mathcal{K}$, the ideal of compact operators, the isomorphism given by the vector representation or by any of the regular representations (together with $\mathcal{G}_x \cong M$). If M is a discrete set with k elements, then $C^*(\mathcal{G}) \cong M_k(\mathbb{C})$ and the convolution product is given by matrix multiplication.

Example 3.10 (Transformation (or Action) groupoid). Suppose that a Lie group G acts on the smooth manifold M from the right. The *transformation groupoid* over $M \times \{e\} \cong M$, denoted by $M \rtimes G$, is the set $M \times G$ with structure maps $d(m, g) = (m \cdot g, e)$, $r(m, g) = (m, e)$, $(m, g)(m \cdot g, h) = (m, gh)$, $u(m, e) = (m, e)$, and $\iota(m, g) = (m \cdot g, g^{-1})$. For more on the action groupoid, one may see [37,53,62].

Example 3.11 (Bundle of Lie groups). If $\mathcal{G} \to M$ is a *bundle of Lie groups*, i.e, $d = r$ (hence each fiber is a Lie group), then $\Psi^m(\mathcal{G})$ consists of smooth families of invariant and properly supported pseudodifferential operators on the fibers of $\mathcal{G} \to M$. Clearly, vector bundles are a special case of bundle of Lie groups.

4. Double Layer Potentials on Plane Sectors

We consider a plane sector $\Omega_\theta := \{r\alpha : r \in (0, \infty), \alpha \in (0, \theta)\}$ with angle θ. Thus, the boundary $\partial\Omega_\theta$ consists of two rays, which we label as L_1 and L_2, respectively.

4.1. The Double Layer Potential Operator Associated with a Plane Sector

Before we calculate the explicit form of the double layer potential operator associated to the Laplace operator and Ω_θ, we recall the definition of the Mellin convolution operator and the definition of the Mellin transform [61].

Definition 4.1. Let $p = p(r) \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ and $u \in \mathcal{C}_c^\infty(\mathbb{R}^+)$. Define the function Pu on $\mathbb{R}^+$ by

$$Pu(r) = p * u(r) = \int_0^\infty p(r/s)\, u(s)\, \frac{ds}{s}.$$

The operator P will be called the smoothing Mellin convolution operator on $\mathbb{R}^+$ with convolution kernel p.

Definition 4.2. Let p be the convolution kernel of a smoothing Mellin convolution operator P on $\mathbb{R}^+$. The Mellin transform $\mathcal{M}p$ of p is defined by

$$\mathcal{M}p(t) = q(t) = \int_0^\infty s^{-it} p(s) \frac{ds}{s}.$$

We recall the following standard properties of the Mellin transform [61].

Proposition 4.3. *Suppose P is a smoothing Mellin convolution operator with convolution kernel p. Then for any $u \in \mathcal{C}_c(\mathbb{R}^+)$, we have*

$$\mathcal{M}(p * u)(t) = \mathcal{M}(Pu)(t) = \mathcal{M}p(t)\mathcal{M}u(t).$$

Remark 4.4. Some general references on Mellin convolution operators and the Mellin transform in solving boundary value problems, include Kapanadze and Schulze [26], Egorov and Schulze [17], Lewis and Parenti [35], Melrose [45,46], Schrohe and Schulze [64,65], Schulze [66].

The double layer potential with a function ϕ on $\partial\Omega_\theta$ is defined by

$$(K_\theta \phi)(x) := -\frac{1}{\pi}\int_{\partial\Omega_\theta} \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\phi(y)d\sigma(y), \tag{3}$$

where $x, y \in \partial\Omega$, $\nu(y)$ is the exterior unit normal to a point $y \in \partial\Omega_\theta$. So K_θ depends on the locations of x and y on the boundary. We further define for $i = 1, 2$,

$$\phi_i(x) := \phi(x) \quad \text{for } x \in L_i, \qquad \phi_i(x) := 0, \quad \text{otherwise.}$$

Denote by

$$(K_{ij}\phi_i)(x) := -\frac{1}{\pi}\int_{L_j} \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\phi_i(y)d\sigma(y), \qquad j = 1, 2.$$

Note that if x, y belong to the same ray, then $x - y$ is perpendicular to $\nu(y)$, i.e., $(x-y)\cdot\mu(y) = 0$. Then, it is clear that the operator K_θ can be represented as a 2×2 matrix, i.e.,

$$K_\theta \phi = \begin{pmatrix} 0 & K_{12} \\ K_{21} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Let w and z be the points on L_1 and L_2, respectively. A direct calculation leads to

$$\begin{aligned}(K_{12}\phi_2)(w) &= \int_0^\infty k_\theta(w/z)\phi_2(z)\frac{dz}{z},\\ (K_{21}\phi_1)(z) &= \int_0^\infty k_\theta(z/w)\phi_1(w)\frac{dw}{w},\end{aligned}$$

where

$$k_\theta(r) = \frac{1}{\pi}\frac{r\sin\theta}{r^2 + 1 - 2r\cos\theta}. \tag{4}$$

It is clear that K_{12} and K_{21} are both Mellin convolution operators with the same kernel k_θ. For simplicity, in the text below, we let $k = k_\theta$. Then, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ and $r \in \mathbb{R}^+$, we define

$$(\widetilde{K}\varphi)(r) = \int_0^\infty k(r/s)\varphi(s)\frac{ds}{s}. \tag{5}$$

Therefore, the operator $\widetilde{K}$ is a convolution operator, and $K_{12} = K_{21} = \widetilde{K}$ on $\mathcal{C}_c^\infty(\mathbb{R}^+)$.

We need the following lemma.

Lemma 4.5. *For each $\xi \in \mathbb{R}$, define*

$$f(\xi) = \int_{-\infty}^{\infty} \frac{\cos(x\xi)}{e^x - 2\cos\theta + e^{-x}}\, dx.$$

Then we have

$$f(\xi) = \frac{\pi}{\sin\theta}\left(\frac{e^{(2\pi-\theta)\xi} - e^{\theta\xi}}{e^{2\pi\xi} - 1}\right). \tag{6}$$

Moreover, for all $\xi \in \mathbb{R}$, we have $0 < f(\xi) \leqslant \dfrac{\pi}{|\sin\theta|}$.

Proof. Since $f(\xi)$ is an even function, we can suppose that ξ is positive. It is easy to see that

$$f(\xi) = \int_{-\infty}^{\infty} \frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta} dz,$$

We choose the contour $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \bigcup \Gamma_4$, where $\Gamma_1 = \{(x,0)| - M \leqslant x \leqslant M\}$, $\Gamma_2 = \{(M, iy)|0 \leqslant y \leqslant M\}$, $\Gamma_3 = \{(x, Mi)| - M \leqslant x \leqslant M\}$, and $\Gamma_4 = \{(-M, iy)|0 \leqslant y \leqslant M\}$, for M large enough.

On Γ_2, we have

$$\begin{aligned}\int_{\Gamma_2}\left|\frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}dz\right| &= \int_0^M \frac{e^{-y\xi}}{|e^{M+iy} + e^{-M-iy} - 2\cos\theta|}dy \\ &\leqslant \int_0^M \frac{1}{e^M - e^{-M} - 2}dy \\ &\leqslant \frac{M}{e^M - e^{-M} - 2}.\end{aligned}$$

Hence we see that as $M \to \infty$,

$$\int_{\Gamma_2} \frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}dz \to 0.$$

For the same reason, as $M \to \infty$, we have

$$\int_{\Gamma_4} \frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}dz \to 0.$$

On Γ_3, if we take $M_k = 2k\pi$, then we have

$$\begin{aligned}\int_{\Gamma_3}\left|\frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}dz\right| &= \int_{-M_k}^{M_k} \frac{e^{-M_k\xi}}{|e^{x+iM_k} + e^{-x-iM_k} - 2\cos\theta|}dx \\ &\leqslant \int_{-M_k}^{M_k} \frac{e^{-M_k\xi}}{|e^x + e^{-x} - 2\cos\theta|}dx \\ &\leqslant \int_{-\infty}^{\infty} \frac{e^{-M_k\xi}}{|e^x + e^{-x} - 2\cos\theta|}dx \\ &\leqslant Ce^{-M_k\xi},\end{aligned}$$

where C is a constant. Thus as $M_k \to \infty$, we have

$$\int_{\Gamma_3} \frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}dz \to 0.$$

Now let us find the singularities of the integrand, i.e., the roots of the equation

$$e^z + e^{-z} - 2\cos\theta = 0.$$

So we get $z = (2k\pi \pm \theta)i$, where $k = 0, \pm 1, \pm 2, \ldots$. In the interior of Γ, we see that

$$z = \theta i, (2\pi \pm \theta)i, (4\pi \pm \theta)i, \ldots.$$

Let

$$g(z) = \frac{e^{iz\xi}}{e^z + e^{-z} - 2\cos\theta}.$$

Next let us compute the residue of $g(z)$ at each pole. It is clear that each singularity is simple. Therefore, we calculate

$$\begin{aligned} Res(g, (2k\pi \pm \theta)i) &= \lim_{z\to(2k\pi\pm\theta)i} \frac{e^{iz\xi}(z - (2k\pi \pm \theta)i)}{e^z + e^{-z} - 2\cos\theta} \\ &= \frac{e^{-(2k\pi\pm\theta)\xi}}{2i\sin(2k\pi \pm \theta)} \\ &= \frac{e^{-(2k\pi\pm\theta)\xi}}{2i\sin(\pm\theta)} \end{aligned}$$

Therefore, the Residue Theorem allows us to compute

$$\begin{aligned} f(\xi) &= \frac{\pi}{\sin\theta}\left(e^{-\theta\xi} + \sum_{k=1}^{\infty}\left(e^{-(2k\pi+\theta)\xi} - e^{-(2k\pi-\theta)\xi}\right)\right) \\ &= \frac{\pi}{\sin\theta}\left(e^{-\theta\xi} + \frac{e^{-(2\pi+\theta)\xi}}{1 - e^{-2\pi\xi}} - \frac{e^{-(2\pi-\theta)\xi}}{1 - e^{-2\pi\xi}}\right) \\ &= \frac{\pi}{\sin\theta}\left(e^{-\theta\xi} - \frac{e^{\theta\xi} - e^{-\theta\xi}}{e^{2\pi\xi} - 1}\right) \\ &= \frac{\pi}{\sin\theta}\left(\frac{e^{(2\pi-\theta)\xi} - e^{\theta\xi}}{e^{2\pi\xi} - 1}\right), \end{aligned}$$

where we use the fact that the series is absolutely convergent. This proves (6).

Define

$$\varphi(\theta) = e^{-\theta\xi} - \frac{e^{\theta\xi} - e^{-\theta\xi}}{e^{2\pi\xi} - 1}.$$

Then we have $\varphi(0) = 1$, $\varphi(\pi) = 0$, and $\varphi(2\pi) = -1$. Moreover, we compute

$$\varphi'(\theta) = -\xi e^{-\theta\xi} - \frac{\xi e^{\theta\xi} + \xi e^{-\theta\xi}}{e^{2\pi\xi} - 1} < 0 \text{ for any positive } \xi > 0.$$

This implies that $\varphi(\theta) > 0$ for all $\theta \in (0, \pi)$ and $\varphi(\theta) < 0$ for all $\theta \in (\pi, 2\pi)$. As a consequence, we have $0 < f(\xi) \leqslant \dfrac{\pi}{|\sin\theta|}$ for any positive ξ.

Moreover, we have

$$f(0) = \lim_{\xi\to 0} f(\xi) = \frac{\pi}{\sin\theta}\cdot\frac{\pi - \theta}{\pi} = \frac{\pi - \theta}{\sin\theta} > 0.$$

Since f is even, we see that $0 < f(\xi) \leqslant \dfrac{\pi}{|\sin\theta|}$ for all $\xi \in \mathbb{R}$. $\square$

Remark 4.6. By the above lemma, the Mellin transform of $k(r)$ in (4) can be computed as follows

$$\begin{aligned}
\mathcal{M}k(t) &= \frac{1}{\pi}\int_0^\infty s^{-it}\frac{s\sin\theta}{s^2+1-2s\cos\theta}\frac{ds}{s}\\
&= \frac{1}{\pi}\int_{-\infty}^\infty e^{-itx}\frac{e^x\sin\theta}{e^{2x}+1-2e^x\cos\theta}dx\\
&= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\cos(xt)\sin\theta}{e^x+e^{-x}-2\cos\theta}dx\\
&= \frac{\sin\theta}{\pi}\frac{\pi}{\sin\theta}\left(e^{-\theta t}-\frac{e^{\theta t}-e^{-\theta t}}{e^{2\pi t}-1}\right)\\
&= \frac{e^{(2\pi-\theta)t}-e^{\theta t}}{e^{2\pi t}-1}.
\end{aligned}$$

Note that the function

$$\mathcal{M}k(z) = \frac{e^{(2\pi-\theta)z}-e^{\theta z}}{e^{2\pi z}-1}$$

is holomorphic in the strip $\{z \in \mathbb{C} : -1 < \Im(z) < 1\}$, where $\Im(z)$ is the imaginary part of z.

Let M_f denote the multiplication operator by f. Recall the operator $\widetilde{K}$ from Equation (5). Then, $M_{r^a}\widetilde{K}M_{r^{-a}}$ has Mellin convolution kernel

$$k_a(r) = \frac{1}{\pi}\cdot\frac{r^{a+1}\sin\theta}{r^2-2r\cos\theta+1}.$$

Notice that $k_a(r)$ is a smooth function on $r > 0$ (provided that $a > -1$). The Mellin transform of k_a is calculated as follows

$$\begin{aligned}
\mathcal{M}k_a(t) &= \frac{e^{(2\pi-\theta)(t-ai)}-e^{\theta(t-ai)}}{e^{2\pi(t-ai)}-1}\\
&= \mathcal{M}k(t-ai).
\end{aligned} \tag{7}$$

The double layer potential operator associated to Ω_θ and the Laplace operator takes the form $K_\theta = \begin{pmatrix} 0 & K_{12}\\ K_{21} & 0\end{pmatrix} = \begin{pmatrix} 0 & \widetilde{K}\\ \widetilde{K} & 0\end{pmatrix}$. Thus, we have

$$K_{\theta,a} := \begin{pmatrix} M_{r^a} & 0\\ 0 & M_{r^a}\end{pmatrix} K_\theta \begin{pmatrix} M_{r^{-a}} & 0\\ 0 & M_{r^{-a}}\end{pmatrix} = \begin{pmatrix} 0 & M_{r^a}\widetilde{K}M_{r^{-a}}\\ M_{r^a}\widetilde{K}M_{r^{-a}} & 0\end{pmatrix}.$$

Let $\widetilde{K_a} := M_{r^a}\widetilde{K}M_{r^{-a}}$. The following theorem is well-known. We give a short proof here.

Theorem 4.7. *The operator $\pm I + K_{\theta,a} := \pm I + M_{r^a}K_\theta M_{r^{-a}}$ is invertible if and only if $1-(\mathcal{M}\widetilde{K_a}(t))^2 \neq 0$ for all $t \in \mathbb{R}$, where $\mathcal{M}\widetilde{K_a}$ is the Mellin Transform of $\widetilde{K_a}$.*

Proof. The identity operator I has convolution kernel $\delta(x/y)$ and the Mellin transform of the δ-function is the function of constant 1. By Proposition 4.3, we know that the Mellin transform of a Mellin convolution operator is the multiplication operator by the Mellin transform of the kernel function. Clearly, $\widetilde{K_a}$ is a Mellin convolution operator. Then the theorem follows. $\square$

Recall Proposition 2.2. We have, for all $m \in \mathbb{Z}$,

$$\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega_\theta) \cong H^m(\partial\Omega_\theta, g),$$

where the metric $g = r^{-2}g_e$.

We define a_θ by

$$a_\theta = \min\{\frac{\pi}{\theta}, \frac{\pi}{2\pi-\theta}\} = \begin{cases} \pi/(2\pi-\theta), & 0<\theta<\pi, \\ \pi/\theta, & \pi<\theta<2\pi. \end{cases}$$

Theorem 4.8. *If $a \in (-a_\theta, a_\theta)$, then $\pm I + M_{r^a}K_\theta M_{r^{-a}}$ is invertible on $\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega_\theta)$, that is,*

$$\pm I + K_\theta : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega_\theta) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega_\theta)$$

are invertible.

Proof. Recall the Mellin transform of k_a in Eq. (7). By Theorem 4.7, we need to find the (positive) smallest $a > 0$ such that $\mathcal{M}k_a(t) = \pm 1$ for some $t \in \mathbb{R}$.

Therefore, we compute

$$\begin{aligned} & e^{(2\pi-\theta)(t-ai)} - e^{\theta(t-ai)} = \pm(e^{2\pi(t-ai)} - 1) \\ \iff & e^{(2\pi-\theta)(t-ai)} \pm 1 = e^{\theta(t-ai)} \pm e^{2\pi(t-ai)} \\ \iff & e^{(2\pi-\theta)(t-ai)} \pm 1 = e^{\theta(t-ai)}(1 \pm e^{(2\pi-\theta)(t-ai)}) \end{aligned}$$

For case "+", we have $e^{\theta(t-ai)} = 1$ or $e^{(2\pi-\theta)(t-ai)} = -1$.
For case "−", we have $e^{\theta(t-ai)} = -1$ or $e^{(2\pi-\theta)(t-ai)} = 1$.
Hence we obtain

$$\begin{aligned} & \theta t - a\theta i = k\pi i \quad \text{or} \quad (2\pi-\theta)t - (2\pi-\theta)ai = k\pi i,\ k = \pm1, \pm2, \ldots \\ \iff & t = 0 \quad \text{and} \quad a = \frac{k\pi}{\theta} \quad \text{or} \quad a = \frac{k\pi}{2\pi-\theta},\ k = \pm1, \pm2, \cdots. \end{aligned}$$

Hence the (positive) smallest a would be $a_\theta = \min\{\frac{\pi}{\theta}, \frac{\pi}{2\pi-\theta}\}$. $\square$

Remark 4.9. The interior and exterior Dirichlet problems correspond to the operators $I+K_\theta$ and $-I+K_\theta$, respectively. The above calculation shows that the interior and exterior Dirichlet problems are indistinguishable when we use the double layer potentials. So we should consider the operators $\pm I + K_\theta$ at the same time.

The following proposition is needed, which gives the explicit description of the function in the kernel of $\pm I + K_\theta$.

Proposition 4.10. *Suppose that a function u satisfies that $(\pm I + K_\theta)u = 0$, then u is of the form*

$$u(r) = \sum_i c_i r^{a_i}$$

where $c_i's$ are constants, and $a_i \in \{\frac{k\pi}{\theta}, \frac{k\pi}{2\pi-\theta}, |\ k = \pm 1, \pm 2, \dots\}$

Proof. Since u satisfies $(I + K_\theta)u = 0$, multiplying both sides by r^a from the left gives

$$r^a u + r^a K_\theta r^{-a}(r^a u) = 0$$

Hence we have

$$(I + r^a K_\theta r^{-a})(r^a u) = 0.$$

By Theorem 4.7 and the above calculation in Theorem 4.8, we know that

$$\mathcal{M}k_a(t) = \pm 1 \iff t = 0 \quad \text{and} \quad a = \frac{k\pi}{\theta}, \quad \frac{k\pi}{2\pi - \theta}, \ k = \pm 1, \pm 2, \dots.$$

Therefore, the Mellin transform of the function $r^a u$ has support only at $t = 0$, so u is a linear combination of $r^{a_i}s$. The case for $-I + K_\theta$ is the same. □

4.2. Relations to Lie Groupoids

We are in position to identify the double layer potential operator K with a smooth invariant family of operators on some Lie groupoid.

Let $\widetilde{\mathcal{H}} = [0, \infty] \rtimes \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$ is regarded as a commutative group. So $\widetilde{\mathcal{H}}$ is an action groupoid. It is easy to see that

$$C^*(\widetilde{\mathcal{H}}) = C([0, \infty]) \rtimes \mathbb{R}^+.$$

Notice that $C[0, \infty]$ is a unital commutative C^*-algebra and $C^*(\widetilde{\mathcal{H}})$ is not unital. Moreover, we have

$$C_0(\mathbb{R}^+) = C^*(\mathbb{R}^+) \subset C^*(\widetilde{\mathcal{H}}).$$

Next we would like to define an (order $-\infty$) invariant family P on the groupoid $\widetilde{\mathcal{H}} = [0, \infty] \rtimes (0, \infty)$, such that $\pi(P) = \widetilde{K}$, where $\widetilde{K}$ is defined by Eq. (5) and π is the vector representation of $\Psi^\infty(\widetilde{\mathcal{H}})$ on $\mathcal{C}_c^\infty(0, \infty)$ uniquely determined by

$$(\pi(P)f) \circ r = P(f \circ r).$$

We notice that $[0, \infty]$ is the space of units and $(0, \infty)$ is an invariant open dense subset of the compact space $[0, \infty]$. Then $\pi(\Psi^\infty(\widetilde{\mathcal{H}}))$ maps $\mathbb{C}_c^\infty(0, \infty)$ to itself.

We define a map $\phi_{x_0} : \mathbb{R}^+ \to \widetilde{\mathcal{H}}_{x_0}$ by

$$\phi_{x_0}(x) = (x_0 x, x^{-1}).$$

It is easy to see that the map ϕ_{x_0} is a diffeomorphism for all $x_0 \in [0, \infty]$. So we can use this map ϕ_{x_0} to identify $\mathbb{R}^+$ and $\widetilde{\mathcal{H}}_{x_0}$.

For any $f(x) \in \mathcal{C}_c^\infty(0, \infty)$, we define

$$F(x, y) = f(y^{-1}), \quad \forall (x, y) \in \widetilde{\mathcal{H}}.$$

Clearly the function $F(x, y)$ is smooth on $\widetilde{\mathcal{H}}$. Furthermore, if we restrict the function F to $\widetilde{\mathcal{H}}_{x_0}$, we get $F|_{\widetilde{\mathcal{H}}_{x_0}} = F(x_0 x, x^{-1})$. Hence we obtain in this way a smooth function on $\mathcal{C}_c^\infty(\widetilde{\mathcal{H}}_{x_0})$. On the other hand, any smooth function on $\widetilde{\mathcal{H}}_{x_0}$ can be written (by using ϕ_{x_0}) in the form $g(x_0x, x^{-1})$. Finally, if $x_0 \in [0,\infty]$, we have a one-to-one correspondence between $\mathcal{C}_c^\infty(0,\infty)$ and $\mathcal{C}_c^\infty(\widetilde{\mathcal{H}}_{x_0})$ in the following way:

$$f \in \mathcal{C}_c^\infty(0,\infty) \leftrightarrow F|_{\widetilde{\mathcal{H}}_{x_0}} = F(x_0x, x^{-1}) \in \mathcal{C}_c^\infty(\widetilde{\mathcal{H}}_{x_0}).$$

Suppose that $\tilde{p}(x,y)$ is a smooth function on $(0,\infty)\times(0,\infty)$. We can define an integral operator on $\mathcal{C}_c^\infty(0,\infty)$ by

$$(\widetilde{P}f)(x) = \int_0^\infty \tilde{p}(x,y)f(y)\frac{dy}{y}, \qquad \forall f \in \mathcal{C}_c^\infty(0,\infty).$$

Then we define $p : \bigcup\limits_{x_0\in(0,\infty)} \widetilde{\mathcal{H}}_{x_0} \times \widetilde{\mathcal{H}}_{x_0} \to \mathbb{R}$ by

$$p|_{\widetilde{\mathcal{H}}_{x_0}\times\widetilde{\mathcal{H}}_{x_0}} : p((x_0x, x^{-1}), (x_0y, y^{-1})) = \tilde{p}(x_0x, x_0y),$$

where we use the map ϕ_{x_0} to identify $(0,\infty)$ and $\widetilde{\mathcal{H}}_{x_0}$.

We define a family of integral operators $P = (P_{x_0})$, where $P_{x_0} : \mathcal{C}_c^\infty(\widetilde{\mathcal{H}}_{x_0}) \to \mathcal{C}^\infty(\widetilde{\mathcal{H}}_{x_0})$, $x_0 \in (0,\infty)$, given by

$$\begin{aligned}(P_{x_0}F)(x_0x, x^{-1}) &= (P_{x_0}f)(x)\\ &= \int_0^\infty \tilde{p}(x_0x, x_0y)f(y)\frac{dy}{y},\end{aligned}$$

where $f(x) = (F\circ\phi_{x_0})(x)$.

Lemma 4.11. *The family of integral operators $P = (P_{x_0})$, $x_0 \in (0,\infty)$, is invariant.*

Proof. For fixed $x_1, x_2 \in (0,\infty)$, there exists a unique element $g = (x_2, x_2^{-1}x_1) \in \widetilde{\mathcal{H}}$ such that $d(g) = d(x_2, x_2^{-1}x_1) = x_1$ and $r(g) = r(x_2, x_2^{-1}x_1) = x_2$. Suppose we have $F \in \mathcal{C}_c^\infty(\widetilde{\mathcal{H}}_{x_1})$. Then F can be written as $F(x_1x, x^{-1}) = f(x)$, where $f \in \mathcal{C}_c^\infty(0,\infty)$. So we have

$$\begin{aligned}(U_gF)(x_2x, x^{-1}) &= F\left((x_2x, x^{-1})(x_2, x_2^{-1}x_1)\right)\\ &= F(x_2x, x_2^{-1}x_1x^{-1}),\end{aligned}$$

therefore,

$$(P_{x_2}U_gF)(x_2x, x^{-1}) = \int_0^\infty \tilde{p}(x_2x, x_2y)f(x_1^{-1}x_2y)\frac{dy}{y}.$$

On the other hand, we obtain

$$(P_{x_1}F)(x_1x, x^{-1}) = \int_0^\infty \tilde{p}(x_1x, x_1y)f(y)\frac{dy}{y}.$$

Let $h(x_1x, x^{-1}) = (P_{x_1}F)(x_1x, x^{-1})$. Thus

$$\begin{aligned}(U_gh)(x_2x, x^{-1}) &= h(x_2x, x_2^{-1}x_1x^{-1})\\ &= \int_0^\infty \tilde{p}(x_2x, x_1y)f(y)\frac{dy}{y}\\ &= \int_0^\infty \tilde{p}(x_2x, x_2z)f(x_1^{-1}x_2z)\frac{dz}{z},\end{aligned}$$

where we replace x with $x_1^{-1}x_2x$ in (3.1) and substitute x_2z for x_1y. Hence

$$P_{x_2}U_g = U_gP_{x_1}.$$

This shows that P is invariant. $\square$

Remark 4.12. For an invariant family $P = (P_{x_0})$, $x_0 \in (0, \infty)$, if we take the limit as $x_0 \to 0$, then we obtain that P_0 is an integral operator with kernel

$$p_0(x, y) = \lim_{x_0 \to 0} \tilde{p}(x_0x, x_0y).$$

For instance, if $\tilde{p}(x, y) = a(x)f(xy^{-1})$, then P_0 has integral kernel $a(0)f(xy^{-1})$.

Proposition 4.13. *We have $\pi(P) = \widetilde{P}$, where π is the vector representation of $\Psi^\infty(\widetilde{\mathcal{H}})$ on $\mathcal{C}_c^\infty(0, \infty)$.*

Proof. For all $f \in \mathcal{C}_c^\infty(0, \infty)$, we have

$$(\widetilde{P}f) \circ r(x_0x, x^{-1}) = \int_0^\infty \tilde{p}(x_0x, y)f(y)\frac{dy}{y},$$

and

$$\begin{aligned}(\pi(P)f) \circ r(x_0x, x^{-1})) &= P(f \circ r(x_0x, x^{-1}))\\ &= \int_0^\infty \tilde{p}(x_0x, x_0y)(f \circ r \circ \phi_{x_0})(y)\frac{dy}{y}\\ &= \int_0^\infty \tilde{p}(x_0x, x_0y)f(x_0y)\frac{dy}{y}\\ &= \int_0^\infty \tilde{p}(x_0x, z)f(z)\frac{dz}{z}\end{aligned}$$

This implies $\pi(P) = \widetilde{P}$. $\square$

Proposition 4.14. *There exists a unique invariant family $P = (P_{x_0})$, $x_0 \in (0, \infty)$, so that $\pi(P) = \widetilde{K}$, $\lim_{x_0 \to 0} P_{x_0} = \widetilde{K}$ and $\lim_{x_0 \to \infty} P_{x_0} = \widetilde{K}$, where $\widetilde{K}$ is defined in Eq.* (5).

Proof. We simply take $\tilde{p}(t, s)$ to be $k(t/s)$. Then the corresponding family defined above satisfies the requirements. $\square$

We summarize what we have proved in the following theorem.

Theorem 4.15. *The operator $\widetilde{K}$ can be (uniquely) identified with a smoothly invariant family $P = (P_{x_0})$, $x_0 \in [0, \infty]$, on the groupoid $\widetilde{\mathcal{H}} = [0, \infty] \rtimes (0, \infty)$, such that $\pi(P) = \widetilde{K}$, $P_0 = \widetilde{K}$, and $P_\infty = \widetilde{K}$.*

Remark 4.16. Since $\widetilde{K}$ is not uniformly supported, it does not belong to $\Psi^{-\infty}(\widetilde{\mathcal{H}})$ in the sense of [58]. However, it does belong to the pseudodifferental algebra of order $-\infty$ on $\widetilde{\mathcal{H}}$ constructed in [68].

However, since $M_{r^a}\widetilde{K}M_{r^{-a}}$ is a smoothing operator (with smooth kernel) for $a \in (-1, 1)$, we obtain the following mapping property:

Proposition 4.17. *For all $k, l \in \mathbb{Z}$, and $a \in (-1, 1)$, we have*

$$M_{r^a} \widetilde{K} M_{r^{-a}} : H^k(\mathbb{R}^+, g) \to H^l(\mathbb{R}^+, g),$$

where the metric $g = r^{-2} g_e$.

Recall that $\widetilde{\mathcal{H}} := [0, \infty] \rtimes (0, \infty)$, where the action is given by dilation. We have

Proposition 4.18. *If $a \in (-1, 1)$, then we have $M_{r^a} \widetilde{K} M_{r^{-a}} \in C^*(\widetilde{\mathcal{H}})$.*

Proof. The kernel of $M_{r^a} \widetilde{K} M_{r^{-a}}$ is

$$\widetilde{k}_a(x, s) = k_a(x/s) = \frac{1}{\pi} \frac{x^{1+a} s^{1-a} \sin\theta}{x^2 + s^2 - 2xs\cos\theta}.$$

Thus, it suffices to show that $\widetilde{k}_a(x, s)$ belongs to $L^1(\widetilde{\mathcal{H}})$, that is, $||\widetilde{k}_a||_I < \infty$, where $||\cdot||_I$ is defined in Sect. 3. Indeed, we have

$$\begin{aligned}
||\widetilde{k}_a||_{I,d} &= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_0^\infty \frac{(xs^{-1})^{1+a}\, s^{1-a} \sin\theta}{(xs^{-1})^2 + s^2 - 2(xs^{-1})s\cos\theta} \frac{ds}{s} \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_0^\infty \frac{x^{1+a} s^{2-2a} \sin\theta}{x^2 + s^4 - 2xs^2\cos\theta} \frac{ds}{s} \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{x^{1+a} e^{2y(1-a)} \sin\theta}{x^2 + e^{4y} - 2xe^{2y}\cos\theta}\, dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{(xe^{-2y})^a\, (xe^{2y}) \sin\theta}{x^2 + e^{4y} - 2xe^{2y}\cos\theta}\, dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{(xe^{-2y})^a \sin\theta}{xe^{-2y} + x^{-1}e^{2y} - 2\cos\theta}\, dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{e^{az} \sin\theta}{2(e^z + e^{-z} - 2\cos\theta)}\, dz \\
&< \infty.
\end{aligned}$$

Hence, $||\widetilde{k}_a||_{I,d}$ is independent of $x \in [0, \infty]$ and it is finite if $-1 < a < 1$. Similarly, we obtain

$$\begin{aligned}
||\widetilde{k}_a||_{I,r} &= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_0^\infty \frac{x^{1+a} s^{1-a} \sin\theta}{x^2 + s^2 - 2xs\cos\theta} \frac{ds}{s} \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{x^{1+a} e^{y(1-a)} \sin\theta}{x^2 + e^{2y} - 2xe^{y}\cos\theta}\, dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{(xe^{-y})^a xe^{y} \sin\theta}{x^2 + e^{2y} - 2xe^{y}\cos\theta} dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{(xe^{-y})^a \sin\theta}{xe^{-y} + x^{-1}e^{y} - 2\cos\theta} dy \\
&= \frac{1}{\pi} \sup_{x \in [0,\infty]} \int_{-\infty}^\infty \frac{e^{az} \sin\theta}{e^z + e^{-z} - 2\cos\theta}\, dz \\
&< \infty.
\end{aligned}$$

Thus, if $-1 < a < 1$, then $||\widetilde{k}_a||_{I,r}$ is also finite and independent of $x \in [0, \infty]$. As a consequence, $||\widetilde{k}_a||_I$ is finite, hence we have $M_{r^a}\widetilde{K}M_{r^{-a}} \in C^*(\widetilde{\mathcal{H}})$. □

Recall that K_θ is the double layer potential operator associated to the plane sector Ω_θ, and K_θ is a 2×2 matrix with diagonal 0 and off diagonal $\widetilde{K}$. Denote by $\mathcal{P}_2$ the pair groupoid of the set $\{1, 2\}$. According to the above discussion, we can identify K_θ with a (smooth) invariant family of pseudodifferential operators on the Lie groupod $\widetilde{\mathcal{H}} \times \mathcal{P}_2$ which satisfies some requirements.

Clearly, we can apply the same argument to $M_{r^a}K_\theta M_{r^{-a}}$. So we summarize the results in the following proposition.

Proposition 4.19. *Let $a \in (-1, 1)$.*

1. *There is a unique smooth invariant family $Q = (Q_{(x,i)})$, $x \in [0, \infty]$ and $i \in \{1, 2\}$ on the Lie groupoid $\widetilde{\mathcal{H}} \times \mathcal{P}_2$, such that $\pi(Q) = K_{\theta,a}$, $Q_{(0,i)} = K_{\theta,a}$, and $Q_{(\infty,i)}$, where $i = 1, 2$ and π is the vector representation.*
2. *We have $M_{r^a}K_\theta M_{r^{-a}} \in C^*(\widetilde{\mathcal{H}}) \otimes M_2(\mathbb{C})$;*
3. *For all $k, l \in \mathbb{Z}$, the following mapping property holds:*
$$M_{r^a}K_\theta M_{r^{-a}} : \mathcal{K}^k_{\frac{1}{2}+a}(\partial\Omega_\theta) \to \mathcal{K}^l_{\frac{1}{2}+a}(\partial\Omega_\theta).$$

5. Double Layer Potentials on Plane Polygons

Throughout this section, we use Ω to denote a simply connected polygon in $\mathbb{R}^2$ with n successive vertices. We label these vertices as $P_1, P_2, \ldots, P_n, P_{n+1} = P_1$, the angle at vertex P_i as θ_i, and still denote by K the double layer potential operator associated to Ω and the Laplace operator Δ. To get the Fredholmness property and invertibility of the operator $\pm I + K$ on some weighted Sobolev spaces on Ω, we need some C^*-algebra knowledge and results in [11].

Motivated by the study of boundary value problem on Ω (in the present paper, the domain Ω is assumed not to have ramified cracks), Carvalho and Qiao associated to Ω a (natural) Lie groupoid $\mathcal{G}$ [11]. Let us briefly review the construction in that paper. Denote by $\mathcal{H} = [0, \infty) \rtimes (0, \infty)$ the action groupoid. For each angle θ_i, we can associate the Lie groupoid $\mathcal{J}_i = \mathcal{H} \times \mathcal{P}_2$. Let $M_0 := \partial\Omega \backslash \{P_1, P_2, \ldots, P_n\}$ and $M_0^2 = M_0 \times M_0$ be the pair groupoid of M_0. Then we can glue $J_i's$ and M_0^2 in a certain way to obtain the Lie groupoid $\mathcal{G}$.

Let r_Ω be the (smoothened) distance function constructed in [7]. Recall that $\Psi^m(\mathcal{G})$ denotes the pseudodifferential operators of order m on $\mathcal{G}$ and by $C^*(\mathcal{G})$ the C^*-algebra of the Lie groupoid $\mathcal{G}$ which is called layer potentials C^*-algebra in [11].

Proposition 5.1. *If $a \in (-1/2, 1/2)$, then $M_{r_\Omega^a}KM_{r_\Omega^{-a}} \in C^*(\mathcal{G})$.*

Proof. Because the restriction of the double layer potential operator K (associated to Ω) to angle θ_i, is just K_{θ_i} which is a (smoothing) Mellin convolution operator discussed in Sect. 4, and so is $M_{r_\Omega^a}KM_{r_\Omega^{-a}}$ for $a \in (-1, 1)$. Hence, we can identify $M_{r_\Omega^a}KM_{r_\Omega^{-a}}$ with a unique smooth invariant family of operators on the Lie groupoid $\mathcal{G}$ such that the vector representation of the family at each angle is $K_{\theta,a}$ (Sect. 4).

According to the paper of Lewis and Parenti [35], the double layer potential operator K may be represented as an $n \times n$ matrix $[K_{i,j}]_{i,j=1}^{n}$, and $K_{i,j}$ maps the functions on jth side to the function on ith side, and involves three possibilities: zero ($i = j$), K_{θ_i} (if ith side and jth side do touch), and $K_{i,j}$ (if ith side and jth side do not touch).

1. If ith side and jth side are adjacent, then $K_{i,j}$ corresponds to K_{θ_i}. Thus, by Proposition 4.19, we have $M_{r_\Omega^a} K_{\theta_i} M_{r_\Omega^{-a}}$ belongs to $C^*(\mathcal{G})$ for $|a| < 1$.
2. If ith side and jth side do not touch, we need to consider the interaction among non-adjacent sides. In this case, since they are non-adjacent, the kernel of $K_{i,j}$ is bounded. However, we are using the metric $g = r_\Omega^{-2} g_e$ to identify weighted Sobolev spaces and usual Sobole spaces. As a consequence, to show that
$$M_{r_\Omega^a} K_{i,j} M_{r_\Omega^{-a}} : \mathcal{K}^0_{\frac{1}{2}}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}}(\partial\Omega)$$
is compact (hence belongs to the C^*-algebra $C^*(\mathcal{G})$), it suffices to show that $M_{r_\Omega^a} K_{i,j} M_{r_\Omega^{-a}}$ is a Hilbert–Schmidt operator on $L^2(\partial\Omega)$, which requires that r_Ω^{-a} be square-integrable near 0, i.e., $a < 1/2$. By symmetry, the function r_Ω^a should be square-integrable near 0 as well, i.e., $a > -1/2$. Hence, we obtain that $|a| < 1/2$. $\square$

Recall that by Proposition 2.3, we have the identification $\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega) \simeq H^m(\partial\Omega, g)$, where the metric $g = r_\Omega^{-2} g_e$, and g_e is the Euclidean metric. By the definition of weighted Sobolev spaces, we have $\mathcal{K}^0_{\frac{1}{2}}(\partial\Omega) \simeq r_\Omega^{\frac{1}{2}} L^2(\partial\Omega)$.

Let $\theta_0 := \min\{\frac{\pi}{\theta_1}, \frac{\pi}{2\pi-\theta_1}, \frac{\pi}{\theta_2}, \frac{\pi}{2\pi-\theta_2}, \dots, \frac{\pi}{\theta_n}, \frac{\pi}{2\pi-\theta_n}\}$. It is clear that
$$1/2 < \theta_0 < 1.$$

Proposition 5.2. *Let Ω be a simply connected polygon on $\mathbb{R}^2$, and K be the double layer potential operator associated to Ω and the Laplace operator Δ. Then for $a \in (-\theta_0, 1/2)$, the operators*
$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega)$$
are both Fredholm.

Proof. First of all, let us assume that $a \in (-1/2, 1/2)$. By Corollary 6.4 in [11] and Proposition 5.1, it is sufficient to prove that $\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ is elliptic and the restriction of $\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ to each angle θ_i is invertible. The ellipticity of $\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ is clear, and the invertibility of $\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ (restricted to angle θ_i) is proved in Theorem 4.8. Therefore, we establish that for $a \in (-1/2, 1/2)$,
$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega)$$
are Fredholm.

Secondly, we have the identification $\mathcal{K}^0_0(\partial\Omega) \cong L^2(\partial\Omega)$. In [35], Lewis and Parenti showed that $\pm I + K : L^2(\partial\Omega) \to L^2(\partial\Omega)$ are isomorphisms. In particular, $\pm I + K$ are Fredholm operators. Hence the family of operators $\pm I + M_{r_\Omega^b} K M_{r_\Omega^{-b}}$ acting on $L^2(\partial\Omega)$ is still Fredholm for $|b| < \epsilon$ for some $\epsilon > 0$ small enough. In view of Proposition 4.10, we can take $\epsilon = \theta_0 - 1/2 > 0$.

Combining the above results, we obtain that for $-\theta_0 < a < 1/2$,

$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega)$$

are Fredholm. $\square$

Lemma 5.3. *Let Ω be a polygon on $\mathbb{R}^2$, and K be the double layer potential operator associated to Ω and the Laplace operator Δ. The operators*

$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega)$$

are injective for all $a \in (-\theta_0, 1/2)$.

Proof. In Proposition 4.10, we find all the possible singular values for double layer potentials at each vertex. The range of a in the lemma excludes all these values. Thus the conclusion holds for $a \in (-\theta_0, 1/2)$. $\square$

Theorem 5.4. *Let Ω be a simply connected polygon on $\mathbb{R}^2$, and K be the double layer potential operator associated to Ω and the Laplace operator Δ. The operators*

$$\pm I + K : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$$

are both isomorphisms for all $a \in (-\theta_0, 1/2)$.

Proof. As in [7], the family of operators

$$\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}} : \mathcal{K}^0_{\frac{1}{2}}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}}(\partial\Omega)$$

depends continuously on a. In [35], Lewis and Parenti already proved that $\pm I + K : L^2(\partial\Omega) \to L^2(\partial\Omega)$ are isomorphisms. By identification $\mathcal{K}^0_0(\partial\Omega) \cong L^2(\partial\Omega)$, we see that for $a_0 = -\frac{1}{2} \in (-\theta_0, 1/2)$,

$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a_0}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a_0}(\partial\Omega)$$

are isomorphisms. Then by Proposition 5.2, we know that for all $a \in (-\theta_0, 1/2)$, the operators

$$\pm I + K : \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^0_{\frac{1}{2}+a}(\partial\Omega)$$

are Fredholm of index zero. Moreover, since the operators $M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ (regarded as a smooth invariant family of operators on the Lie groupoid $\mathcal{G}$) are a smooth family of operators with smooth kernels, the indices of the operators

$$\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}} : \mathcal{K}^m_{\frac{1}{2}}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}}(\partial\Omega)$$

are independent of m. As a result, the operators $\pm I + M_{r_\Omega^a} K M_{r_\Omega^{-a}}$ are all Fredholm of index zero. Then the desired result is followed by Lemma 5.3. $\square$

Remark 5.5. For interior and exterior Neumann problems, we need to solve the boundary integral equations $-I + K^*$ and $I + K^*$ (choosing a suitable fundamental solution of the Laplace operator), respectively, where $K^*(x, y) = K(y, x)$ [21,35,69]. Thus, it is possible to extend our method to Neumann boundary value problems.

6. Conclusion

In [11], to a plane polygon Ω we associate a boundary groupoid $\mathcal{G}$ with the space of units given by a desingularization M of $\partial\Omega$. The layer potentials C^*-algebra associated to Ω is defined to be the groupoid convolution algebra $C^*(\mathcal{G})$.

In the present paper, we apply pseudodifferential operator (on Lie groupoids) techniques to the method of layer potentials to solve Dirichlet boundary value problems for the Laplace operator on a simply connected plane polygon.

More precisely, let Ω be a simply connected plane polygon with vertices $P_1, P_2, \ldots, P_n$. Denote by θ_i the interior angle at vertex P_i. The main ingredients of our proofs are as follows:

1. For each (infinite) plane sector Ω_{θ_i}, $i = 1, 2, \ldots, n$, since the double layer potential operator K_{θ_i} (associated to Ω_{θ_i} and the Laplace operator Δ) is a Mellin convolution operator, we use the Mellin transform to prove that the operator K_{θ_i} is invertible for suitable weighted Sobolev spaces on $\partial\Omega_{\theta_i}$.
2. Using the properties of K_{θ_i} and some results of Lewis and Parenti, we show that the double layer potential operator K (associated to $\partial\Omega$ and the Laplace operator Δ) belongs to the groupoid convolution algebra $C^*(\mathcal{G})$.
3. Combining the invertibility of K_{θ_i} and general results on pseudodifferential operators on Lie groupoids, we establish the Freholmness of the double layer potential operator K between appropriate weighted Sobolev spaces on $\partial\Omega$.
4. We apply techniques from Melin transform to prove our main theorem. Namely, the operators $\pm I + K$ are in fact isomorphic between suitable weighted Sobolev spaces on $\partial\Omega$. By the result in [7, Theroem 5.9], this implies a solvability result in weighted Sobolev spaces for the Dirichlet problem on Ω.

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